Applications and Uses for the Maxwell Garnett Mixing Formula

In the last lecture we saw derivations of the Maxwell Garnett and Bruggeman mixing formulas. We also saw how these formulas can be obtained from a common heterogenous mixing equation proposed by Aspnes.

Our objective in this lecture is to provide examples showing the use and accuracy one can expect from using the formulas. While these two formulas can be considered as arising from a common formula (ala Aspnes), they are applicable to completely different types of mixtures:

- Maxwell Garnett:
  \[
  \frac{E_{eff} - E_e}{E_{eff} + 2E_e} = f_a \frac{E_a - E_e}{E_a + 2E_e} \quad (1)
  \]

  Works well for regular lattices of particles. \(E_{eff}\) is fairly well predicted for volume fractions to 15%-20%, and varied particle shapes, not just spheres.

  Amazingly, in the case of cubes (or squares in 2-D) the MG formula nearly exactly predicts \(E_{eff}\).

- (Symmetrical) Bruggeman:
  \[
  f_a \frac{E_a - E_{eff}}{E_a + 2E_{eff}} + f_b \frac{E_b - E_{eff}}{E_b + 2E_{eff}} = 0 \quad (2)
  \]
This formula works well with random distributions of particles. As with meo, works best with relatively small volume fraction of particles.

We'll consider separately a number of applications with supporting data for these two mixing formulas.

**Applications of the Maxwell-Garnett Formula**

A good place to start with accessing the use of the M-G formula is with spherical particles in a regular lattice. We will show results for the low frequency \( \varepsilon_{\text{eff}} \) for SC, BCC, FCC lattices of PEC spheres computed a number of ways:

- Maxwell–Garnett mixing formula.
- \( S \) parameter extraction for plane wave - plane slab scattering, MNS.
- McPhedran & McKenzie: accurate, numerical calculation involving spherical harmonics of particles.

The first case we will show is a SC lattice of PEC spheres. Results are from a Year 2 report on an NSF grant by Wykes (PI), Amert (co-PI), and Amagnan (co-PI).

Figure 1.2 Extracted dielectric constant for a simple cubic lattice of perfectly conducting metallic sphere as the volume fraction changes compared with published results.

Figure 1.3 Difference between our extracted dielectric constant and McPhedran's published results for a simple cubic lattice of perfectly conducting metallic spheres as the volume fraction increases.

MG gives very good results up to 30% volume fraction or better.

This lattice is also diamagnetic. The effective μe is shown next, where the MG formula for μe is

$$\mu_{eff} = \frac{1 - \eta_g}{1 + \frac{\eta_g}{2}}$$  \hspace{1cm} (3)
Figure 1.4 Extracted relative permeability for a simple cubic lattice of perfectly conducting metallic sphere as the volume fraction changes.

Figure 1.5 Simulation setup and unit cell of a body centered cubic lattice of perfectly conducting metallic spheres.

Table 1.1 Extracted and published body centered cubic lattice material parameters.

<table>
<thead>
<tr>
<th>Volume Fraction</th>
<th>Published $\varepsilon$ [6]</th>
<th>Extracted $\varepsilon$</th>
<th>Analytical $\mu$</th>
<th>Extracted $\mu$</th>
<th>$\varepsilon$ Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.050</td>
<td>1.158</td>
<td>1.158</td>
<td>0.927</td>
<td>0.921</td>
<td>0.000</td>
</tr>
<tr>
<td>0.100</td>
<td>1.333</td>
<td>1.331</td>
<td>0.857</td>
<td>0.853</td>
<td>0.002</td>
</tr>
<tr>
<td>0.150</td>
<td>1.530</td>
<td>1.532</td>
<td>0.791</td>
<td>0.783</td>
<td>0.002</td>
</tr>
<tr>
<td>0.200</td>
<td>1.751</td>
<td>1.753</td>
<td>0.727</td>
<td>0.722</td>
<td>0.002</td>
</tr>
<tr>
<td>0.250</td>
<td>2.002</td>
<td>2.001</td>
<td>0.667</td>
<td>0.660</td>
<td>0.001</td>
</tr>
<tr>
<td>0.300</td>
<td>2.292</td>
<td>2.292</td>
<td>0.609</td>
<td>0.603</td>
<td>0.000</td>
</tr>
<tr>
<td>0.350</td>
<td>2.631</td>
<td>2.633</td>
<td>0.553</td>
<td>0.544</td>
<td>0.002</td>
</tr>
<tr>
<td>0.400</td>
<td>3.035</td>
<td>3.037</td>
<td>0.500</td>
<td>0.491</td>
<td>0.002</td>
</tr>
<tr>
<td>0.450</td>
<td>3.532</td>
<td>3.531</td>
<td>0.449</td>
<td>0.440</td>
<td>0.001</td>
</tr>
<tr>
<td>0.500</td>
<td>4.166</td>
<td>4.168</td>
<td>0.400</td>
<td>0.389</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Similarly good results with BCC lattice. Alternatively, here's a different method of computation for a BCC lattice of conducting spheres.

Keith W. Whites

FIG. 3. Computed $\varepsilon_{eff}$ for a body-centered-cubic lattice of conducting spheres. The Maxwell Garnett results are from Eq. (29) and the T-matrix results are from Eq. (25) using the solution method described in Sec. II.
As recorded in that paper by Whites in Fig 6, MG went to 50 for PBC lattice up to lattices with 36% volume fraction. Nice!

MG formula can be used for other particle shapes as well, though with mixed results for relatively high volume fraction.

Cubes particles:

Surprisingly, though, is the incredible accuracy of the MG formula for cube particles (or square in 2-D). Here showing results for a S.C. lattice of PBC cube particles.

---

**Fig. 3.** Computed $\varepsilon_r$ or for 3-D lattices of conducting spheres and conducting cubes. Our MM solution for conducting spheres is compared with data from [10]. Note that the vertical $\varepsilon_r$ scale for the sphere and cube data are much different.
Experimental results undertaken in that same paper to verify that phenomenon. (Measuring quasi-static $\sigma_{\text{eff}}$.)

Fig. 6. Photograph of the conducting cube lattice apparatus for measuring quasi-static effective conductivity. The upper and lower plates as well as the cube are brass. The four-sided box is Lexan. For sample #2 shown above, the measured volume fraction was $f = 0.1952 \pm 0.0005$, as listed in Table I.

Fig. 8. Predicted quasi-static effective permittivity and measured effective conductivity for a simple cubic lattice of conducting cubes. The measurements were performed at 80 kHz. Error bars indicate one standard deviation in both $\varepsilon_{\text{eff}}$ and volume fraction.
Because of this remarkable behavior of what we'll call cute media, we've been able to "engineer", or control/design, the effective material properties of different systems.

One example of this is changing the effective direct impedance of resistive films (Kapton NC) by creating square-shaped perforations of various sizes, but in a regular pattern. Can even grade the properties across the film.

[See the Metamaterials 2009 presentation by White and Clover.]