From the vector potential method discussed earlier, the magnetic vector potential $\vec{A}$ was found to satisfy the P.D.E.

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\vec{J}$$

Having the solution:

$$\vec{A}(\vec{r}) = \vec{J}(\vec{r}') \times \phi(\vec{r}, \vec{r}')$$

we can also derive a P.D.E. for the scalar electric potential, $\phi$.

Starting from Maxwell's equations:

$$\nabla \times \vec{E} = -j \omega \mu \vec{H}$$

$$\nabla \times \vec{H} = j \omega \varepsilon \vec{E} + \vec{J}$$

In the usual fashion, define

$$\vec{A} = \nabla \phi$$

$$\vec{E} = -j \omega \mu \vec{A} - \nabla \phi$$

Take $\nabla \cdot \theta \rightarrow \nabla \cdot \vec{E} = -j \omega \mu \nabla \cdot \vec{A} - \nabla \cdot \nabla \phi$

Now take $\nabla \cdot \phi \rightarrow \nabla \cdot \nabla \phi = 0 = j \omega \varepsilon \nabla \cdot \nabla \phi + \nabla \cdot \vec{J}$

Now solve in for $\nabla \cdot \vec{E}$ giving:

$$-j \omega \varepsilon [-j \omega \mu \nabla \cdot \vec{A} - \nabla \cdot \vec{E}] = \nabla \cdot \vec{J}$$
using the continuity eqn : \[ \nabla \cdot \mathbf{F} = -j \omega \mu \mathbf{e} \]

in the Lorentz gauge : \[ \nabla \cdot \mathbf{A} = -j \omega \mu \mathbf{e} \mathbf{A}_e \]

then (3) can be written as

\[ -j \omega \mu (-j \omega \mathbf{e} \mathbf{e}_e) - \nabla^2 \mathbf{e}_e = \frac{\mathbf{P}_e}{\epsilon} \]

or

\[ \nabla^2 \mathbf{e}_e + k^2 \mathbf{e}_e = -\frac{\mathbf{P}_e}{\epsilon} \quad (4) \]

The scalar electric potential must satisfy a P.D.E. of exactly the same form as each Cartesian component of the vector magnetic potential. The only difference is the forcing fit!

Then by analogy, the solutions to (4) are given as

\[ \mathbf{e}_e (F) = \frac{\mathbf{P}_e (F')}{\epsilon} * \phi (F, F') \]

or

\[ \mathbf{e}_e (F) = \int \frac{\mathbf{P}_e (F')}{\epsilon} \frac{e^{-j k |F-F'|}}{4 \pi |F-F'|} \, dV \]

source volume
For quasi-static problems, i.e., as was shown earlier for the point dipole (for \(|k_r| \ll 1\)), the fields have the same form as the static problem — but they vary harmonically with time.

For this problem, then as \(|k_r|\) becomes very small,\[^{2}\]

\[
\nabla^2 \vec{E}_e + k_e^2 \vec{E}_e = -\frac{\vec{P}_e}{\varepsilon}
\]

\[
\nabla \times \vec{E}_e \propto \int \frac{\vec{P}(\vec{r}')}{\varepsilon} \cdot \frac{1}{4\pi |\vec{r}-\vec{r}'|} \, d\vec{r}'
\]

For static (\(\omega = 0\)) problems, this solution is exact!\[^{0}\]
The first problem we’re going to numerically solve is a PEC plate of infinitesimal thickness, having a prescribed potential impressed on it.

As we’ve determined, the potential in space due to an arbitrary distribution of charge density is

$$\Phi_0(F) = \int \frac{\rho_s(F')}{\varepsilon} \frac{1}{4\pi |\mathbf{r}-\mathbf{r}'|} \, dV'$$

For a surface charge density, as we have on a PEC surface, the integral reduces to:

$$\Phi_0 (F) = \int_{s'} \frac{\rho_s(F)}{\varepsilon} \frac{1}{4\pi |\mathbf{r}-\mathbf{r}'|} \, ds'$$  \hspace{1cm} (5)

Remember: our objective is to first find $$\mathbf{E}$$ on the plate (5'). Once that is known, we can find $$\mathbf{E}$$ anywhere from (5) (static).
To find the $\beta$ numerically, we first will apply the boundary conditions imposed on this structure:

B.C.'s: On the plate $1x1s_0$, $l_1x1s_0$, $z=0$

$$E(x) = V$$ for $F \in \{ \text{pts on plate} \}$.

That's it!

So from (5) we obtain an electric potential integral equation (EPIE) for $\beta$ as

$$V = \int_{S'} \frac{\beta(F')}{\varepsilon} \frac{1}{4\pi |F-F'|} \, ds'$$

for $F, F' \in \{ \text{pts on plate} \}$

(Equation (6) is a very difficult equation to solve analytically. In fact, a closed form solution is surely impossible to obtain (for $\beta$). The difficulty, of course, is that our unknown of interest is part of the integrand!)

We will solve eqn (6) numerically using what is typically called the Moment Method (MM).