One of the simplest "radiation" problems is finding the fields produced by an infinite, planar sheet of impressed surface current -

Objective is to solve for the fields produced by this time-harmonic current density -

\[ \mathbf{J} = -x \mathbf{J}_0 \delta(r) \]

Solution methodology -

- Maxwell's eqns.
- Simplify - \( \frac{\partial}{\partial x} \rightarrow 0 \), \( \frac{\partial}{\partial y} \rightarrow 0 \)
- Form wave eqns (PDE's)
- Solve - plane wave solns.
- Apply b.c.'s: (1) Radiation condition 
  (2) Jump discontinuity in \( \mathbf{H} \)
Perhaps one of the simplest "radiation" problems which can be solved analytically is that of an infinite, planar sheet of "impressed" surface current as shown.

This "impressed" current is assumed to be the source of the radiated fields.

Let \( \mathbf{J} = \vec{x} J_0 \cos \omega t = -i \omega J_0 \delta(\vec{x}) \).

The objective is to solve for the fields which are produced by this time-harmonic current density.

\[ \mathbf{J} = \vec{x} J_0 \delta(\vec{x}) \]

Dirac delta function
Starting with Maxwell's equations:

\[ \nabla \times \mathbf{E} = -\mu_0 \mathbf{H} \quad \therefore \quad \nabla \times \mathbf{H} = \mathbf{J} + j \omega \mathbf{E} \]

using the constitutive parameters \( \mathbf{E} = \varepsilon_0 \mathbf{E} \quad \mathbf{H} = \mu_0 \mathbf{H} \)

then

\[ \nabla \times \mathbf{E} = -j \mu_0 \omega \mathbf{H} \quad \nabla \times \mathbf{H} = \mathbf{J} + j \omega \varepsilon_0 \mathbf{E} \]

Away from the source and in free space, \( \mathbf{J} = 0 \). Also, since we have an infinite current sheet, we can simplify the problem considerably.

\[ \nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \]

There is no variation in either the geometry or the source in the \( x \) and \( y \) directions so that \( \frac{\partial}{\partial x} \to 0, \frac{\partial}{\partial y} \to 0 \).

\[ \therefore \quad \nabla \times \mathbf{E} = \hat{x} \left( -\frac{\partial E_y}{\partial z} \right) - \hat{y} \left( \frac{\partial E_z}{\partial z} \right) \]

Similarly

\[ \nabla \times \mathbf{H} = \hat{x} \left( -\frac{\partial H_y}{\partial z} \right) - \hat{y} \left( \frac{\partial H_z}{\partial z} \right) \]
Substituting into Eqn. (1) →

\[-\omega_0^2 \frac{\partial^2 E_y}{\partial t^2} + \frac{\partial E_y}{\partial x} = -j \omega \mu_0 \frac{\partial H_z}{\partial t}\]
\[-\omega_0^2 \frac{\partial^2 E_x}{\partial t^2} + \frac{\partial E_x}{\partial t} = j \omega \varepsilon_0 E\]

The objective now is to combine these equations in such a manner so that the two resulting scalar equations involve only one unknown function.

To do this, take \( \frac{\partial^2}{\partial t^2} \) of the first eqn. and multiply second by \( j \omega \mu_0 \), giving

\[-\omega_0^2 \frac{\partial^2 E_y}{\partial t^2} + \frac{\partial E_y}{\partial x} = -j \omega \mu_0 \left( \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} \right).
\]
\[-\omega_0^2 \frac{\partial^2 E_x}{\partial t^2} + \frac{\partial E_x}{\partial t} = -j \omega \varepsilon_0 \left( \frac{\partial^2 E_x}{\partial t^2} + \frac{\partial^2 E_y}{\partial t^2} + \frac{\partial^2 E_z}{\partial t^2} \right).
\]

Substituting from the second equation into the first and equating vector components,

\[-\omega_0^2 \frac{\partial^2 E_y}{\partial t^2} = \omega \varepsilon_0 \mu_0 E_x
\]
\[-\omega_0^2 \frac{\partial^2 E_x}{\partial t^2} = \omega \varepsilon_0 \mu_0 E_y
\]

or

\[\frac{\partial^2 E_y}{\partial t^2} + k_y^2 E_y = 0\]
\[\frac{\partial^2 E_x}{\partial t^2} + k_x^2 E_x = 0\]

where \( k_x^2 = \omega^2 \varepsilon_0 \mu_0 \) (2)
Equations (2) are known as wave equations since their solutions give rise to fields which vary in space (and time) as waves.

Their solutions are of the form:

\[ E_y = A e^{-j k z} + B e^{j k z} \]
\[ E_x = C e^{-j k z} + D e^{j k z} \]

in each region 1 and 2. The unknown constants \( A, B, C, \) and \( D \) can be determined by applying the boundary conditions.

First B.C.: \( \hat{n} \times (E_y - E_x) = 0 \) \( \text{at} \ z = 0 \)
where \( \hat{n} = \hat{z} \)

With the current sheet \( \sigma \) \( z = 0 \), we can rationalize the fact that in region 1 \( (z < 0) \) there exists only an outgoing wave of the form \( e^{-j k z} \) and in region 2 \( (z > 0) \) there also exists an outgoing wave of the form \( e^{j k z} \). This rationalization is related to the reflection condition or alternatively to causality.
The total fields in each region are then:

\[ E_x = \hat{x} D \epsilon \hat{r} + \hat{y} B \hat{z} \]
\[ E_z = \hat{x} C \hat{r} + \hat{y} A \hat{z} \]
\[ \hat{z} \times [ (\hat{x} C + \hat{y} A) - (\hat{x} D + \hat{y} B) ] = 0 \]

or \[ \frac{\partial B}{\partial t} \hat{z} + \frac{\partial D}{\partial t} \hat{x} = 0 \] (Faraday's law is continuous)

The other B.C. is \[ \hat{x} \times (A_1 - A_2) = \hat{J}_S \]

The \( J \) field in each region may be obtained from \( E \) and Maxwell's equations since

\[ \nabla \times E = -j \mu_0 \dot{H} \quad \text{where} \quad \nabla \times E = \begin{vmatrix} x & y & z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix} = -x \left( \frac{\partial E_y}{\partial z} \right) - y \left( \frac{\partial E_x}{\partial z} \right) \]

\[ -j \mu_0 \dot{H} = -\hat{x} \frac{\partial E_x}{\partial z} + \hat{y} \frac{\partial E_y}{\partial z} \]

\[ H = \hat{x} \frac{1}{\mu_0} \frac{\partial E_y}{\partial z} - \hat{y} \frac{1}{\mu_0} \frac{\partial E_x}{\partial z} \]
\[
\begin{align*}
\text{in region 2:} & \quad H_z = \hat{z} A \frac{j k_0}{\omega \mu_0} C - \hat{y} \frac{k_0}{\omega \mu_0} C e^{-j k_0 z} \\
\text{in region 1:} & \quad H_z = \hat{z} A \frac{j k_0}{\omega \mu_0} C - \hat{y} \frac{k_0}{\omega \mu_0} C e^{-j k_0 z} \\
\theta & = 0, \quad \frac{\partial}{\partial x}(H_y - H_z) = -\hat{x} J_0 \\
\text{Giving:} & \\
- \hat{y} A \frac{k_0}{\omega \mu_0} - \hat{x} \frac{k_0}{\omega \mu_0} C - \left[ \frac{\hat{y} A k_0}{\omega \mu_0} + \hat{x} \frac{k_0}{\omega \mu_0} C \right] &= -\hat{x} J_0 \\
or / \\
\hat{x}: & \quad C - C = \frac{1}{k_0} \frac{\omega J_0}{\mu_0} \Rightarrow C = \frac{\omega J_0}{k_0} \mu_0 J_0 \\
\hat{y}: & \quad -A \frac{k_0}{\omega \mu_0} - A \frac{k_0}{\omega \mu_0} = 0 \Rightarrow A = 0 \\
\text{Substituting back gives the solutions for the total} \\
\text{fields in each region to be:} & \\
(2.30) & \quad E_z = \hat{z} \frac{\omega J_0}{2 k_0} I_0 \frac{e^{+j k_0 z}}{e^{+j k_0 z}} \quad \quad H_z = \hat{z} \frac{J_0}{2} e^{+j k_0 z} \\
(2.30) & \quad E_z = \hat{x} \frac{\omega J_0}{2 k_0} I_0 \frac{e^{-j k_0 z}}{e^{-j k_0 z}} \quad \quad H_z = \hat{x} \frac{J_0}{2} e^{-j k_0 z} \\
\end{align*}
\]
These results are the complete solutions to the fields which are in space due to the infinite current sheet source.

We notice from the above solutions that $\overline{E}$ and $\overline{H}$ are perpendicular to one another. Also, in planes perpendicular to the direction of propagation ($\pm z$) the fields are constant, or "uniform," in value and direction. Therefore, these fields are called uniform plane waves.

Another characteristic property of this type of wave is the ratio of the orthogonal components of $\overline{E}$ : $\overline{H}$, namely,

- in region 1: $\frac{E_{x1}}{H_{z1}} = \frac{\lambda_0}{2 \mu_0 r_0} = \eta_0$
- in region 2: $\frac{E_{x2}}{H_{z2}} = -\eta_0$

where $\eta_0 = \sqrt{\frac{2\pi}{e_0}}$ and is the intrinsic impedance of the medium.
Summarizing the characteristics of uniform plane waves:

- \( \mathbf{E} = \mathbf{H} \)
- \( \frac{E_x}{H_z} = \pm \eta_0 \)
- The fields have constant direction, magnitude, and phase in planes perpendicular to the direction of propagation (both \( \mathbf{E} || \mathbf{H} \)).