We will shortly require, in our numerical solution methods, the ability to find the fields produced by a given distribution of electric current density: 

**Source/field relationships**

Assuming the source is varying in a time-harmonic fashion with $e^{j\omega t}$, then Maxwell's equations may be written as

\begin{align*}
\nabla \times \mathbf{E} &= -j\omega \mu \mathbf{H} \\
\nabla \times \mathbf{H} &= j\omega \varepsilon \mathbf{E} + j \mathbf{J} \quad \tag{1}
\end{align*}

The objective is to solve for the fields produced by this current distribution $\mathbf{J}$. The solutions, as we shall see, may be expressed as a convolution integral.

**Method 1: The Potential Method**

Taking $\nabla \cdot (1) \Rightarrow \nabla \cdot (\nabla \times \mathbf{E}) = -j\omega \mu \nabla \cdot \mathbf{H}$ (homogeneous)

\[ \mathbf{0} = \nabla \cdot \mathbf{H} \quad \tag{2} \]

This type of field is called solenoidal.

A solenoidal field may be expressed as the curl of another vector (Helmholtz Theorem).
\[ H = \nabla \times A \quad (3) \]

Substituting this result into (1) gives
\[ \nabla \times E = -j \omega M \nabla \times \vec{A} \]

or
\[ \nabla [\vec{E} + j \omega M \vec{A}] = 0 \]

A vector field which is curl-free is called irrotational or conservative and can be expressed as the gradient of another scalar potential
\[ \vec{E} + j \omega M \vec{A} = -\nabla \Phi \quad \uparrow \]
by convention

(4)

Now, substitute both (4) and (3) into (2) giving
\[ \nabla \times \nabla \vec{A} = j \omega [\nabla (\vec{E} + j \omega M \vec{A})] + \vec{J} \]

Vector identity
\[ \nabla \times \nabla \vec{A} = \nabla (\nabla \vec{A}) - \nabla^2 \vec{A} \]

[where \( \nabla^2 \vec{A} \) is defined on the Cartesian components of \( \vec{A} \) only! \( (\nabla \vec{A}, \nabla^2 \vec{A}, \nabla^4 \vec{A}) \)]

\[ \nabla (\vec{E} + j \omega M \vec{A}) - \nabla^2 \vec{A} - \omega \mu \varepsilon \vec{A} + j \omega \varepsilon \vec{E} = \vec{J} \]

Rewriting
\[ \nabla^2 \vec{A} + k^2 \vec{A} - \nabla (\nabla \cdot \vec{A} + j \omega \varepsilon \vec{E}) = -\vec{J} \]
By the Helmholtz theorem, a vector field is uniquely specified (within an additive constant) when both the curl and divergence are specified.

- We've selected $\nabla \times \vec{A} = \vec{A}$

- Now choose $\nabla \cdot \vec{A} = -j\omega \vec{E}$ to simplify the last eqn.

This choice is called the Lorentz Gauge.
For time-varying fields it is consistent with conservation of charge. [$\nabla \cdot \vec{A} = 0 \Rightarrow$ Coulomb Gauge]

Substituting this last choice for $\nabla \cdot \vec{A}$ gives:

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\vec{J} \quad (5)$$

A vector Helmholtz eqn. for $\vec{A}$. It is a set of 3 scalar equations - inhomogeneous O.D.E.'s

Once this equation has been solved for $\vec{A}$, then the fields are given as:

- $\vec{B} = \nabla \times \vec{A}$
- $\vec{E} = -j\mu_0 \vec{H} - \nabla \Phi_0$ or $\omega \vec{E} = -j\omega \vec{A}$
- $\vec{E} = -j\omega \vec{A} + \frac{1}{j\omega} \nabla \cdot \vec{A}$
Method 2: The Direct Method

\[ \nabla \times \mathbf{E} = -j \omega \mu \mathbf{H} \quad (1) \quad \text{and} \quad \nabla \times \mathbf{H} = j \omega \epsilon \mathbf{E} + \mathbf{J} \quad (2) \]

Take \[ \nabla \times (1) \rightarrow \nabla \times \nabla \times \mathbf{E} = -j \omega \mu \nabla \times \mathbf{H} \]

Sub (2) into this giving \[ \nabla \times \nabla \times \mathbf{E} = -j \omega \mu \left( j \omega \epsilon \mathbf{E} + \mathbf{J} \right) \]

or \[ \nabla \left( \nabla \cdot \mathbf{E} \right) - \nabla^{2} \mathbf{E} - \omega^{2} \epsilon \mathbf{E} = -j \omega \mu \mathbf{J} \quad (6) \]

But what is \[ \nabla \cdot \mathbf{E} ? \] Take (\nabla \cdot \mathbf{v} ) of (2)

\[ \nabla \cdot \mathbf{E} = -j \omega \epsilon \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{J} \]

or \[ \nabla \cdot \mathbf{E} = - \frac{1}{j \omega \epsilon} \nabla \cdot \mathbf{J} \]

Sub. into (6) gives

\[ - \frac{1}{j \omega \epsilon} \nabla \cdot \mathbf{J} - \nabla^{2} \mathbf{E} - \kappa^{2} \mathbf{E} = -j \omega \mu \mathbf{J} \]

or \[ \nabla^{2} \mathbf{E} + \kappa^{2} \mathbf{E} = j \omega \mu \mathbf{J} - \frac{1}{j \omega \epsilon} \nabla \cdot \mathbf{J} \quad (7) \]

Once \( \mathbf{E} \) has been solved, can find \( \mathbf{H} \) from Maxwell's equations.
In many other EM courses, primarily undergraduate, this is about the extent of the source/field concept.

Typically at this point, the current density $\vec{J}$ would be assumed to have some functional form from which the fields could mathematically be computed using the potential method (5) and Maxwell's equations.

However, in this course we will be solving for either $\vec{A}$ or $\vec{E}$! They are the unknowns.

Once they have been found (either $\vec{A}$ or $\vec{E}$, depending on the problem), we can calculate all other fields, power, etc...

**Problem:** From (5) and (7) we don't have expressions of the form $\vec{A} = (...)$ or $\vec{E} = (...)$. Instead we have

$$ (\nabla^2 + k^2) \vec{A} = -\vec{J} $$

$$ (\nabla^2 + k^2) \vec{E} = -\frac{\nabla \times \vec{J} + k^2 \vec{E}}{j\omega} $$

Form of L.H.S is the same, forcing terms are different.

(3) To actually solve for $\vec{A}$ or $\vec{E}$ we must invert this linear operator $\nabla^2 + k^2$. 