Lecture 6: The Smith Chart.

The Smith chart began its existence as a very useful graphical calculator for the analysis and design of TLs. It was developed by Phillip H. Smith in the 1930s.

The Smith chart remains a useful tool today to visualize the results of TL analysis, oftentimes combined with computer analysis and visualization as an aid in design.

The development of the Smith chart is based on the normalized TL impedance $\tilde{z}(z)$ defined as

$$\tilde{z}(z) \equiv \frac{Z(z)}{Z_0} = \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

(1)

where $Z(z) = V(z)/I(z)$ is the total TL impedance at $z$ and

$$\Gamma(z) = \Gamma_L e^{j2\beta z}$$

(2)

is the generalized reflection coefficient at $z$.

The real and imaginary parts of the generalized reflection coefficient $\Gamma(z)$ will be defined as $\Gamma(z) \equiv \Gamma_r(z) + j\Gamma_i(z)$. Substituting this definition into (1) gives

$$\tilde{z}(z) = \frac{1+(\Gamma_r + j\Gamma_i)}{1-(\Gamma_r + j\Gamma_i)}$$

(3)

Now, we will define $\tilde{z}(z) \equiv r + jx$ and separate (3) into its real and imaginary parts
\[ \tilde{z}(z) \equiv r + jx = \frac{1 + (\Gamma_r + j\Gamma_i) \cdot 1 - (\Gamma_r + j\Gamma_i)^*}{1 - (\Gamma_r + j\Gamma_i) \cdot 1 - (\Gamma_r + j\Gamma_i)^*} \]

\[ = \frac{1 + j2\Gamma_i - (\Gamma_r^2 + \Gamma_i^2)}{1 - 2\Gamma_r + \Gamma_r^2 + \Gamma_i^2} \]

Equating the real and imaginary parts of this last equation gives

\[ r = \frac{1 - (\Gamma_r^2 + \Gamma_i^2)}{(\Gamma_r - 1)^2 + \Gamma_i^2} \quad \text{and} \quad x = \frac{2\Gamma_i}{(\Gamma_r - 1)^2 + \Gamma_i^2} \quad (2.55) \]

Rearranging both of these leads us to the final two equations

\[ \left( \Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \left( \frac{1}{1+r} \right)^2 \quad (2.56a),(4) \]

and

\[ (\Gamma_r - 1)^2 + \left( \Gamma_i - \frac{1}{x} \right)^2 = \left( \frac{1}{x} \right)^2 \quad (2.56b),(5) \]

We will use (4) and (5) to construct the Smith chart.

**Definition:** The Smith chart is a plot of normalized TL resistance and reactance functions drawn in the complex, generalized reflection coefficient \([\Gamma(z)]\) plane.

To understand this, first notice that in the \(\Gamma_r-\Gamma_i\) plane:

1. Equation (4) has only \(r\) as a parameter and (5) has only \(x\) as a parameter.

2. Both (4) and (5) are families of circles.

Consequently, we can plot (4) and (5) in the \(\Gamma_r-\Gamma_i\) plane while keeping either \(r\) or \(x\) constant, as appropriate.
Plot (4) in the $\Gamma_r-\Gamma_i$ plane:

- For $r = 0$: $\Gamma_r^2 + \Gamma_i^2 = 1^2$
- For $r = 1$: $\left(\Gamma_r - \frac{1}{2}\right)^2 + \Gamma_i^2 = \left(\frac{1}{2}\right)^2$
- For $r = \frac{1}{2}$: $\left(\Gamma_r - \frac{1}{3}\right)^2 + \Gamma_i^2 = \left(\frac{2}{3}\right)^2$

Plot these curves in the $\Gamma_r-\Gamma_i$ plane:

![Diagram showing curves in the complex plane](image)

Plot (5) in the $\Gamma_r-\Gamma_i$ plane:

- For $x = 1$: $(\Gamma_r - 1)^2 + (\Gamma_i - 1)^2 = 1^2$
- For $x = -1$: $(\Gamma_r - 1)^2 + (\Gamma_i + 1)^2 = (-1)^2$
- For $x = 100$: $(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{100}\right)^2 = \left(\frac{1}{100}\right)^2$
- For $x = \frac{1}{100}$: $(\Gamma_r - 1)^2 + (\Gamma_i - 100)^2 = 100^2$
Plot these curves in the $\Gamma_r$-$\Gamma_i$ plane:

Combining both of these curves (or “mappings”), as shown on the next page, gives what is called the Smith chart.

One metaphor for these circles of constant TL resistance and constant TL reactance are as “shadows” being projected onto the $\Gamma(z)$ plane. For any TL impedance as specified by an intersection of a constant resistance and a constant reactance circle (the “shadows”), a corresponding value of the TL generalized reflection coefficient at that position on the TL is specified in the complex $\Gamma(z)$ plane.
As quoted from the text (p. 64):

“The real utility of the Smith chart, however, lies in the fact that it can be used to convert from reflection coefficients to
normalized impedances (or admittances), and vice versa, using
the impedance (or admittance) circles printed on the chart.”

Additionally, it is very easy to compute the generalized
reflection coefficient and normalized impedance anywhere on a
homogeneous section of TL.

Notice that the $\Gamma_r$ and $\Gamma_i$ axes are missing from the “combined”
plot. This is also the case for the Smith chart.

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**Important Features of the Smith Chart**

1. By definition $\Gamma(z) = \frac{\bar{z}(z) - 1}{\bar{z}(z) + 1} = \frac{(r + jx) - 1}{(r + jx) + 1}$. Therefore

   $$|\Gamma(z)| = \frac{r + jx - 1}{r + jx + 1} \cdot \frac{r - jx + 1}{r - jx + 1} = \frac{(r - 1)^2 + x^2}{(r + 1)^2 + x^2}$$

   From this result, we can show that if $r \geq 0$ then $|\Gamma(z)| \leq 1$. This condition is met for passive networks (i.e., no amplifiers) and lossless TLs (real $Z_0$).

   Consequently, the standard Smith chart only shows the inside of the unit circle in the $\Gamma_r$-$\Gamma_i$ plane. That is, $|\Gamma(z)| \leq 1$ which is bounded by the $r = 0$ circle described by $\Gamma_r^2 + \Gamma_i^2 = 1$.

2. If $\bar{z}(z)$ is purely real (i.e., $x = 0$), then since
\[ x = \frac{2\Gamma_i}{(\Gamma_r - 1)^2 + \Gamma_i^2} \]

we deduce that \( \Gamma_i = 0 \) (except possibly at \( \Gamma_r = 1 \)).

Consequently, purely real \( \check{z}(z) \) values are mapped to \( \Gamma(z) \) values on the \( \Gamma_r = \Ree[\Gamma(z)] \) axis.

3. If \( \check{z}(z) \) is purely imaginary (i.e., \( r = 0 \)) then from (4)

\[ \Gamma_r^2 + \Gamma_i^2 = 1^2 \]

which is the unit circle in the \( \Gamma_r-\Gamma_i \) plane.

Consequently, purely imaginary \( \check{z}(z) \) values are mapped to \( \Gamma(z) \) values on the unit circle in the \( \Gamma_r-\Gamma_i \) plane.

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**Example N6.1:** Using the Smith chart, determine the voltage reflection coefficient at the load and the TL input impedance.
\[ Z(\lambda = 0) = \frac{Z(\lambda = 0)}{Z_0} = 0.8 + j0.6 \text{ p.u.} \]

\[ \text{mark on Smith chart} \]

- What is \( \Gamma_L \) (the voltage reflection coefficient at the load position, \( \lambda = 0 \))? Read directly from the Smith chart:

\[ |\Gamma_L| = 0.33 \quad \text{and} \quad \angle \Gamma_L = 90^\circ \]

\[ \therefore \Gamma_L = j0.33 \quad (\text{exact soln is } \frac{1}{3}) \]

- For \( \lambda = 0.625 \lambda \), what is \( Z_{\text{in}} \)?

Rotate the \( \Gamma_L \) vector (\( = \Gamma(\lambda = 0) \)) 0.625\( \lambda \)

Towards the generator. From the Smith chart:

Read

\[ Z_{\text{in}} = 2 + j0 \quad \Rightarrow \quad Z_{\text{in}} = Z_{\text{in}} \cdot Z_0 = 2.5 \]

\[ \therefore \quad Z_{\text{in}} = 100.5 \Omega \quad (\text{exact is } 160.5 \Omega) \]
VSWR and the Smith Chart

It was shown in the previous lecture that the voltage magnitude anywhere on the TL can be written as

$$|V(z)| = |V_o^+| |1 + \Gamma(z)|$$  \hspace{1cm} (6)

As derived in the text (Section 2.3)

$$|V(z)|_{\text{max}} = |V_o^+|(1 + |\Gamma_L|)$$

and

$$|V(z)|_{\text{min}} = |V_o^+|(1 - |\Gamma_L|)$$

So, when positioned along the TL at a maximum voltage magnitude:

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_o^+ e^{-j\beta z} \left[1 + \Gamma(z)\right]}{\frac{V_o^+}{Z_0} e^{-j\beta z} \left[1 - \Gamma(z)\right]} = Z_0 \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$  \hspace{1cm} (7)

Using the definition of VSWR from the last lecture

$$\text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$  \hspace{1cm} (8)

then from (7) at a voltage magnitude maximum on the TL

$$Z(z) = Z_0 \cdot \text{VSWR}$$

or

$$\tilde{z}(z) = \text{VSWR}$$  \hspace{1cm} (9)

Because of this last result, we can read the VSWR of a TL directly from the Smith chart.
Similarly, we can show that at a minimum voltage magnitude

\[
\tilde{z}(z) = \frac{1}{\text{VSWR}}
\]  

(10)

In the previous example, we can read VSWR=2 directly from the Smith chart by drawing the constant VSWR circle. This is the circle traced by \( \Gamma(z) \) as \( z \) varies.

However, notice that depending on where we “stop” this rotation of \( \Gamma(z) \) versus \( z \), we obtain different \( \tilde{z}(z) \) values. This happens because \( \Gamma(z) \) is not traversing circles of constant \( r \) and/or \( x \) as \( z \) varies.

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**Smith Admittance Chart**

The Smith chart can be used as an admittance chart as well as an impedance chart.

To see this, recall that we derived the mapping upon which the Smith chart is based \([\tilde{z}(z) \leftrightarrow \Gamma(z)]\) from the normalized TL impedance

\[
\tilde{z}(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}
\]

From this, we can express the normalized TL admittance as
\[ y(z) = \frac{1}{z(z)} = \frac{1 - \Gamma(z)}{1 + \Gamma(z)} \quad (11) \]

We can repeat the construction of the Smith chart with \( y(z) = g + jb \) and \( \Gamma(z) = \Gamma_r + j\Gamma_i \), as we did originally for the impedance chart. Substituting these quantities into (11) we find

\[
\left( \frac{\Gamma_r + \frac{g}{1+g}}{1+g} \right)^2 + \Gamma_i^2 = \left( \frac{1}{1+g} \right)^2 \quad (12)
\]

and

\[
(\Gamma_r + 1)^2 + \left( \Gamma_i + \frac{1}{b} \right)^2 = \left( \frac{1}{b} \right)^2 \quad (13)
\]

A Smith admittance chart can be constructed based on these two equations for circles in the complex \( \Gamma(z) \) plane:

\[ \Gamma_r = \text{Re} \left[ \Gamma(z) \right] \]

\[ \Gamma_i = \text{Im} \left[ \Gamma(z) \right] \]

Notice that circles of constant negative susceptance are in the upper half circle.
This Smith admittance chart looks very similar to the Smith impedance chart. In fact, if we rotated one of these by 180° we obtain the other.

This is actually an easily proved result. Consider the definition of the negative generalized reflection coefficient from (2)

$$-\Gamma(z) = \Gamma_L e^{j(2\beta z + \pi)} = \Gamma_L e^{j \left(2\beta z + \frac{\beta \lambda}{2}\right)}$$

$$= \Gamma_L e^{j \left(2\beta \left(z + \frac{\lambda}{4}\right)\right)}$$

That is,
\[-\Gamma(z) = \Gamma\left(z + \frac{\lambda}{4}\right)\]  

(14)

If we now substitute (14) into (11) we find that

\[y(z) = \frac{1 + \Gamma\left(z + \frac{\lambda}{4}\right)}{1 - \Gamma\left(z + \frac{\lambda}{4}\right)} = \tilde{z}\left(z + \frac{\lambda}{4}\right)\]  

(15)

But what is \(z + \frac{\lambda}{4}\)? It’s a half rotation around the Smith chart.

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**Discussion**

From (15) we can deduce that:

1. If \(\tilde{z}(z)\) is known, then \(y(z)\) is the point on the constant VSWR circle that is diametrically opposite the \(\tilde{z}(z)\) point on the Smith chart. (In this context, remember that a QWT is an impedance inverter device. See Lecture 9.)

2. The Smith chart can be used either as an impedance chart or as an admittance chart. Rather than keeping these two types of charts around, we can use one for either impedance or admittance calculations.

3. One subtlety with these mixed Smith charts is that generalized reflection coefficients are only correctly represented on impedance charts when plotting normalized impedances and on admittance charts when
plotting normalized admittances. You’ll read negative generalized reflection coefficients otherwise (for admittances on impedance charts and impedances on admittance charts).