

The material properties we discussed in the previous lecture pertain to DC and very low frequency fields only.

For time varying fields, the response of a material to an EM field may be quite complex due to the many mechanisms of polarization that are present.

Further, the frequency response of a material can be quite varied due to resonances (over- & under-damped) from the different types of polarization.

It is customary to simultaneously account for polarization & loss effects in a material with complex ϵ & μ as

$$\epsilon \equiv \epsilon' - j\epsilon'' \quad ; \quad \mu \equiv \mu' - j\mu'' \quad (1), (2)$$

Many physical mechanisms may lead to the total values of ϵ' , ϵ'' , μ' , & μ'' . Typically $\epsilon' > 0$, $\epsilon'' > 0$, $\mu' > 0$ & $\mu'' > 0$.

Will illustrate effects of ionic conductivity in a dielectric material. In such a material

$$\bar{J} = \sigma \bar{E} \quad (3)$$

From Ampère's law: $\nabla \times \bar{H} = j\omega \bar{D} + \bar{J}$ (4)

Sub (4) \rightarrow (3) and use constitutive eqn leads to

$$\nabla \times \vec{H} = j\omega \epsilon \vec{E} + \sigma \vec{E} = j\omega \epsilon_0 (\epsilon_r - j \frac{\sigma}{\omega \epsilon_0}) \vec{E} \quad (5)$$

Term in brackets is a relative permittivity where
complex

$$\epsilon_r' = \epsilon_r \quad \text{and} \quad \epsilon_r'' = \frac{\sigma}{\omega \epsilon_0} \quad (6)$$

Single prime term related to polarization, double prime term to loss.

We define the dielectric "loss tangent" $\tan \delta_e$ as the ratio

$$\tan \delta_e \equiv \frac{\epsilon_r''}{\epsilon_r'} = \frac{\sigma}{\omega \epsilon_r' \epsilon_0} \quad \text{for ionic conduction (7)}$$

losses only.

Ref.: Solymar & Walsh.

Frequency Response of Dielectric Materials

The polarization in most materials is due to multiple mechanisms. True for both dielectric & magnetic materials. The contributions each mechanism makes to the total polarization changes with frequency.

An example of this, but for just a hypothetical material, is shown in text fig 1.9.

At low frequencies, all three polarization mechanisms are present: Dipolar (orientational), atomic (molecular), and electronic. At high frequencies (visible light), only the electronic polarization is present - it's the only mechanism that can respond quickly enough to such high frequency fields.

For this hypothetical material, the first two mechanisms "drop out" with increasing frequency. Observe a resonant response at these boundaries. Large losses since ϵ'' is large: similar to accentuated losses in actual ckt's at resonant f's.

One benefit of understanding this frequency response of materials is that we can predict tremendously general information about the material.

For example, we can deduce the electronic contribution to the polarization at microwave freq. by measuring the complex index of refraction at optical frequencies. For this material, most polarization at low f due to dipolar (orientational) polarization. Hence, the material molecules are polar, say for example water. Further, dipolar molecules have permittivity highly dependent on temperature. See the usefulness?

Polar and Non-polar Materials

A polar material is one where the molecules comprising the material have a permanent dipole moment.

Examples include water, methanol, Rutile (a mineral composed of titanium dioxide). From Table 10.1, see that ϵ_r' at low f is much higher than n^2 at optical frequencies. (Water: $\epsilon_r' = 80.4$ while $n^2 = 1.77$.)

A non-polar material is composed of molecules that have no permanent dipole. Si & C (diamond) are examples.

All polarization in non-polar materials is electronic. (?)

Consequently, the index of refraction at optical f 's is equal to $\sqrt{\epsilon_r'}$ at low f . Next!

in-between are materials that are only weakly polar.

Examples are paraffin ($\epsilon_r' = 2.20$, $n^2 = 2.19$) and PTFE ($\epsilon_r' = 2.10$, $n^2 = 1.89$).

Debye Equation

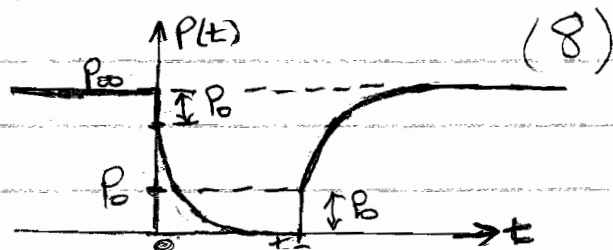
The behavior of polar materials near their resonant frequency is important topic for microwave engineering. This is the frequency regime where this resonance often appears.

The Debye equation is a good model of the frequency response of polar materials in the frequency range where the rotational alignment of the molecules in response to the applied field is encountering a "viscous resistance" to rotation; no longer able to "keep up" with the applied E . Very similar to an overdamped circuit response. The "viscous resistance" is due to a counteracting "force" from thermal buffeting that is working to disorientate the molecules.

Imagine that a steady field is applied to a polar material, and has aligned the molecules. At time $t=0$, the field is removed. The molecules will gradually become randomly orientated from thermal agitation.

In the Debye model, the polarization will decay exponentially as

$$P(t) = (P_0 - P_\infty) e^{-t/\tau}$$



There are probably multiple sources of polarizability in the material. We'll assume that the dipole response is much slower than the response of the electronic and molecular polarizations.

Hence, in the plot there is a sharp decrease (increase) when the applied E is turned off (on).

It can be shown that a material with dipolar polarization as in (8), has a frequency-dependent permittivity of the form

$$\epsilon(\omega) = \epsilon_{\infty} + \frac{\epsilon_s - \epsilon_{\infty}}{1 + j\omega\tau} \quad (9)$$

where $\epsilon_{\infty} = \lim_{\omega \rightarrow \infty} \epsilon(\omega)$ is high freq permittivity (polarization due to mechanisms other than dipoles) and

$\epsilon_s = \lim_{\omega \rightarrow 0^+} \epsilon(\omega)$ is low freq permittivity.

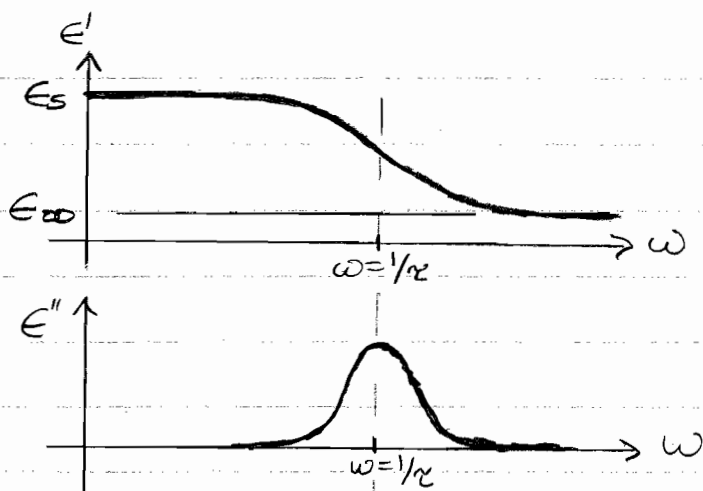
The real and imaginary parts of $\epsilon(\omega)$ are

$$\epsilon'(\omega) = \epsilon_{\infty} + \frac{\epsilon_s - \epsilon_{\infty}}{1 + \omega^2\tau^2} \quad (10)$$

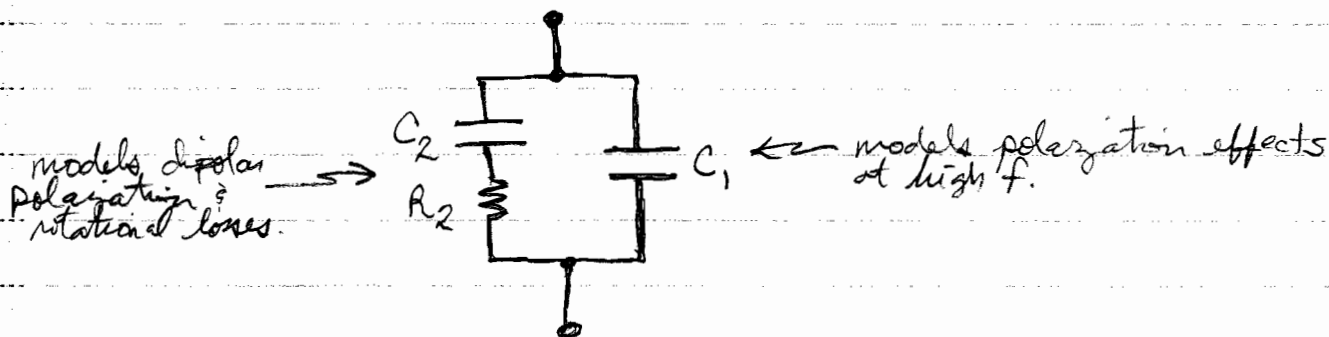
$$\epsilon''(\omega) = (\epsilon_s - \epsilon_{\infty}) \frac{\omega\tau}{1 + \omega^2\tau^2} \quad (11)$$

where $\epsilon(\omega) \equiv \epsilon'(\omega) - j\epsilon''(\omega)$.

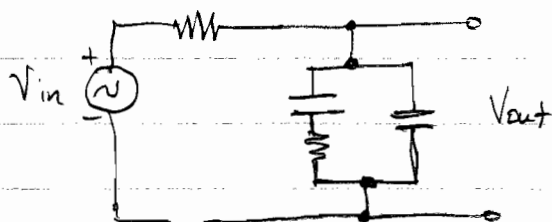
Plot of these quantities vs. ω is:

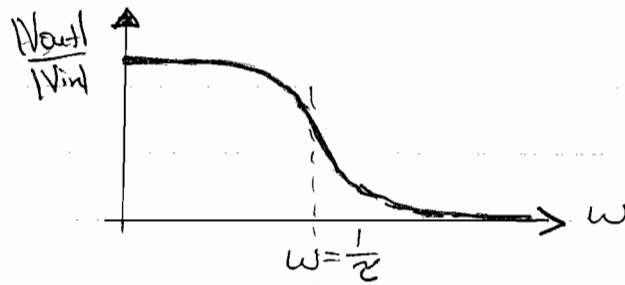


Near $\omega\tau = 1$, the material is said to be "relaxing". This isn't a resonance frequency, but rather a cut-off frequency of sorts. In fact, an equivalent ckt. model associated w/ dipolar polarization is (von Hippel, vol. 2, p. 36):



with source excitation have





low pass filter response can also be thought of as "relaxation". Next.

So what is so special about this relaxation frequency $1/\tau$ for the dipolar material? In this frequency range, the period of the applied electric field is comparable to the alignment time of the molecules in the material. (Ref. Solomon & Walsh, p. 278.)

This Debye model of $\epsilon(\omega)$ in (9) is very important for describing the large permittivity of polar liquids, such as water.

[Fig. 9.15, Bohren & Hoffman]

In 1929, Debye developed classical model of the relaxation time τ of a sphere of radius a in a fluid with viscosity η :

$$\tau = \frac{4\pi\eta a^3}{k_B T} \quad (12)$$

k_B = Boltzmann constant, T = temperature.

The room temperature viscosity at room temp \approx 1 cent Poise (cP)
 where $1 \text{ cP} = 10^{-3} \text{ N}\cdot\text{s}/\text{m}^2$. Assuming $a \approx 10^{-10} \text{ m}$, then

$$\tau \approx \frac{4\pi \cdot 10^{-3} \cdot (10^{-10})^3}{1.38 \times 10^{-23} \text{ J/K} \cdot 293 \text{ K}} \approx 3.1 \text{ ps}$$

This simple model is in reasonable agreement w/ measured value of ~8 ps.

From the phenomenological model in (12) can see how we expect ϵ to vary:

$$\checkmark \text{ as } \eta \uparrow, \tau \uparrow \Rightarrow \epsilon'' \uparrow \text{ , } \epsilon' \omega_R \downarrow$$

$$\checkmark \text{ as } T \downarrow, \tau \uparrow \Rightarrow \text{ " , " (but } \eta = f(T))$$

$$\checkmark \text{ as } a \uparrow, \tau \uparrow \Rightarrow \text{ " , "}$$

Cole-Cole Plot

This is a plot of $\epsilon(\omega)$ in the $\epsilon'' - \epsilon'$ plane as frequency changes. If the material displays dipolar relaxation, this plot will have a very prominent shape: a half circle - or at least a portion of one.

10/15

From (9): $\epsilon' - j\epsilon'' - \epsilon_{\infty} = \frac{\epsilon_s - \epsilon_{\infty}}{1 + j\omega\tau} \Rightarrow |\epsilon' - \epsilon_{\infty} - j\epsilon''|^2 = \left| \frac{\epsilon_s - \epsilon_{\infty}}{1 + j\omega\tau} \right|^2$

s.t. $(\epsilon' - \epsilon_{\infty} - j\epsilon'')(\epsilon' - \epsilon_{\infty} + j\epsilon'') = \frac{(\epsilon_s - \epsilon_{\infty})^2}{(1 + j\omega\tau)(1 - j\omega\tau)}$

or $(\epsilon' - \epsilon_{\infty})^2 + \epsilon''^2 = \frac{(\epsilon_s - \epsilon_{\infty})^2}{1 + \omega^2\tau^2}$ (13)

From (10): $\frac{\epsilon_s - \epsilon_{\infty}}{1 + \omega^2\tau^2} = \epsilon' - \epsilon_{\infty}$. Sub. into (13):

$$(\epsilon' - \epsilon_{\infty})^2 + \epsilon''^2 = (\epsilon' - \epsilon_{\infty})(\epsilon_s - \epsilon_{\infty})$$

Expanding: $\epsilon'^2 - 2\epsilon'\epsilon_{\infty} + \epsilon_{\infty}^2 + \epsilon''^2 = \epsilon'\epsilon_s - \epsilon_{\infty}\epsilon_s - \epsilon'\epsilon_{\infty} + \epsilon_{\infty}^2$

Simplifying: $\epsilon'^2 - \epsilon'(\epsilon_{\infty} + \epsilon_s) + \epsilon''^2 = -\epsilon_{\infty}\epsilon_s$

Add $\left(\frac{\epsilon_s + \epsilon_{\infty}}{2}\right)^2$ to both sides (will complete the square on the LHS):

$$\underbrace{\epsilon'^2 - \epsilon'(\epsilon_{\infty} + \epsilon_s) + \left(\frac{\epsilon_s + \epsilon_{\infty}}{2}\right)^2}_{\left(\epsilon' - \frac{\epsilon_s + \epsilon_{\infty}}{2}\right)^2} + \epsilon''^2 = \left(\frac{\epsilon_s + \epsilon_{\infty}}{2}\right)^2 - \epsilon_{\infty}\epsilon_s$$

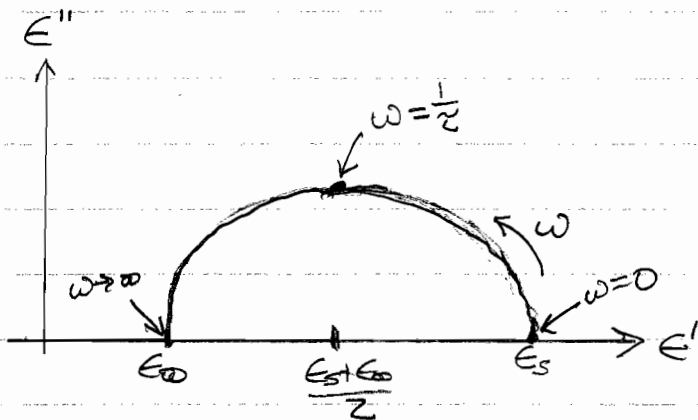
$$\left(\epsilon' - \frac{\epsilon_s + \epsilon_{\infty}}{2}\right)^2 + \epsilon''^2 = \frac{\epsilon_s^2 + 2\epsilon_s\epsilon_{\infty} + \epsilon_{\infty}^2}{4} - \epsilon_{\infty}\epsilon_s$$

or

$$\left(\epsilon' - \frac{\epsilon_s + \epsilon_{\infty}}{2}\right)^2 + \epsilon''^2 = \frac{\epsilon_s^2 - 2\epsilon_s\epsilon_{\infty} + \epsilon_{\infty}^2}{4}$$

$$\therefore \left(\epsilon' - \frac{\epsilon_s + \epsilon_{\infty}}{2}\right)^2 + \epsilon''^2 = \left(\frac{\epsilon_s - \epsilon_{\infty}}{2}\right)^2$$
 (14)

This is the equation for a circle in the $\epsilon' - \epsilon''$ plane:



This is called a Cole-Cole plot. Useful to identify if the material has dipolar polarization. Also useful as a method for extracting ϵ from measurements. Often one will extract ϵ from reflect/trans measurements at each f . Another method is to use a phenomenological model and fit parameters over a freq range. Smooths out errors.

Material may have one or more relaxation frequencies.

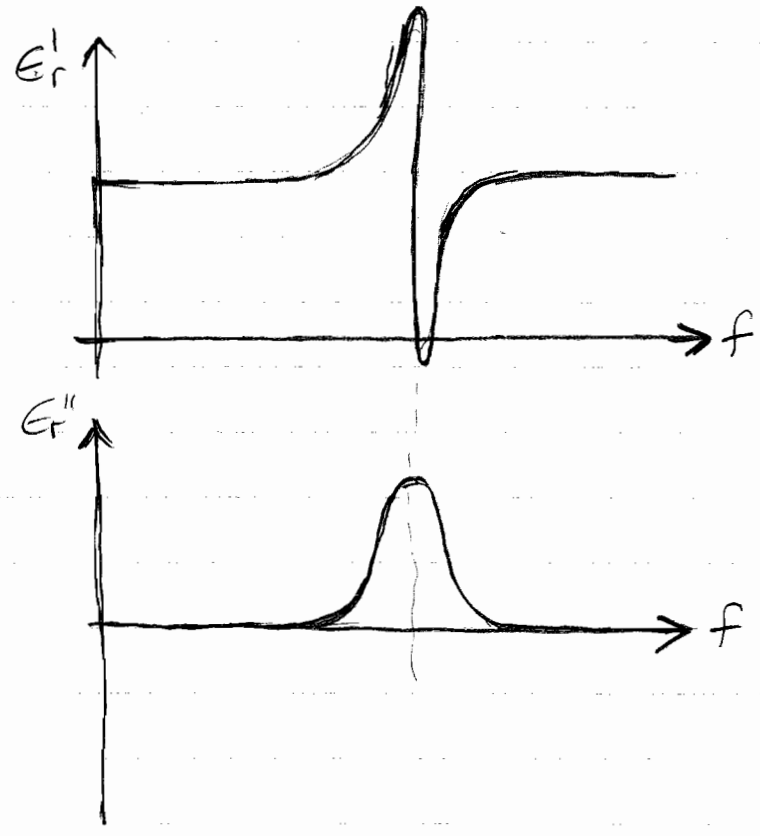
Lorentz Resonances

See the ϵ spectrum in Fig 1.9 for a hypothetical material, we see that the dispersive behavior in ϵ' for atomic & electronic polarization is of a resonant type rather than relaxation.

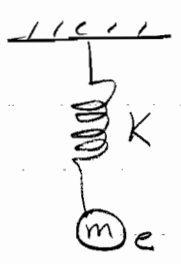
These are Lorentz-type resonances where

$$\epsilon_r = 1 + \chi = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\gamma\omega} \quad (15)$$

where $\omega_p^2 = Ne^2/m\epsilon_0$ $N = \#$ density of electrons m mass m and resonant frequency ω_0 , $\gamma =$ damping factor.



Harmonic oscillator model of material: Electrons and molecular bonds are treated as simple harmonic oscillators, analogous to:



Developed by Lorentz in early 1900's. Same result as if used quantum mechanics

Kramers-Kronig Relations

It turns out for most complex functions describing physical phenomenon that the real and imaginary parts of this fct. cannot be specified independently. Rather, the real and imaginary parts are related!

For complex permittivity, for example, these relationships are stated as:

$$\epsilon_r'(\omega) = 1 + \frac{2}{\pi} \text{P.V.} \int_0^{\infty} \frac{\Omega \epsilon_r''(\Omega)}{\Omega^2 - \omega^2} d\Omega \quad (16)$$

and

$$\epsilon_r''(\omega) = -\frac{2\omega}{\pi} \text{P.V.} \int_0^{\infty} \frac{\epsilon_r'(\Omega) - 1}{\Omega^2 - \omega^2} d\Omega \quad (17)$$

"P.V." means the Cauchy Principle value of the integral, which is necessary to state because of the singularity of the integrals at $\Omega^2 = \omega^2$.

The resistance & reactance of an impedance fct. for a physical device also satisfy^d Kramers-Kronig relation.

Causality is the primary assumption^{in the derivation} leading to the Kramers-Kronig relations, and little more.

The Kramers-Kronig relations (16) & (17) must be satisfied for a causal material. These can be used in some instances to check validity of extracted $\epsilon(\omega)$. Limitation might be a finite freq. range of data.

Ref.: Bohren
& Huffman

Other info. we can glean from (16) & (17):

⊕ If a material has no absorption for all freq., then $\epsilon_r''(\omega) = 0$. From (16) $\Rightarrow \epsilon_r'(\omega) = 1$. This is vacuum. Hence, no causal material is lossless $\forall \omega$. There will be absorption somewhere!

⊕ It is incorrect to state that ϵ_r' is not a fct. of ω while ϵ_r'' is a fct. of ω , and vice versa. From (16) & (17) we see that frequency dependence in one implies frequency dependence in the other.

⊕ The low- and high-frequency behavior of ϵ_r' can be ascertained from (16) regardless of the complexity of ϵ_r'' due to resonances at intermediate frequencies.

As ω becomes very large, the integrand in (16) becomes vanishingly small. Consequently $\epsilon_r' \rightarrow 1$. Makes sense, freq. becomes so high, no polarization mechanism can "keep up" with applied E .

At the opposite end, for $\omega \rightarrow 0^+$ we find from (16) that

$$\epsilon_r'(0^+) = 1 + \frac{2}{\pi} \text{P.V.} \int_0^{\infty} \frac{\epsilon_r''(\Omega)}{\Omega} d\Omega \quad (18)$$

We can understand from (18) that if there is a material w/ static ^{relative} permittivity $\neq 1$, there will be resonance (absorption) bands at some frequencies, with those at relatively lower frequencies contributing more.