Applications and Uses for the Maxwell Garnett Mixing Formula

In the last lecture we saw derivations of the Maxwell Garnett and Bruggeman mixing formulas. We also saw how these formulas can be obtained from a common heterogeneous mixing equation proposed by Aspnes.

Our objective in this lecture is to provide examples showing the use and accuracy one can expect from using the formulas. While these two formulas can be considered as arising from a common formula (ala Aspnes), they are applicable to completely different types of mixtures:

- Maxwell Garnett:

$$\frac{\epsilon_{eff}-\epsilon_e}{\epsilon_{eff}+2\epsilon_e} = f_a \frac{\epsilon_a-\epsilon_e}{\epsilon_a+2\epsilon_e} \qquad (1)$$

Works well for regular lattices of particles. ϵ_{eff} is fairly well predicted for volume fractions to 15%-20%, and varied particle shapes, not just spheres.

Amazingly, in the case of cubes (or squares in 2-D) the MG formula nearly exactly predicts ϵ_{eff}.

- (Symmetrical) Bruggeman:

$$f_a \frac{\epsilon_a-\epsilon_{eff}}{\epsilon_a+2\epsilon_{eff}} + f_b \frac{\epsilon_b-\epsilon_{eff}}{\epsilon_b+2\epsilon_{eff}} = 0 \qquad (2)$$

This formula works well with random distributions of particles. As with MG, works best with relatively small volume fraction of particles.

We'll consider separately a number of applications with supporting data for these two mixing formulas.

Applications of the Maxwell Garnett Formula

A good place to start with accessing the use of the MG formula is with spherical particles in a regular lattice. We will show results for the low frequency ϵ_r & μ_r for SC, BCC, FCC lattices of PEC spheres computed a number of ways:

- Maxwell Garnett mixing formula
- S parameter extraction for plane wave - plane slab scattering. MWS.
- McPhedran & McKenzie: accurate numerical calculation involving spherical harmonics of particles.

The first case we'll show is a S.C. lattice of PEC spheres. Results are from a Year 2 report on an NSF grant by Wilkes (PI), Amert (co-PI), and Anagnostou (co-PI).

McPhedran & McKenzie, Proc. Royal Soc. London, vol. A359, 1978.

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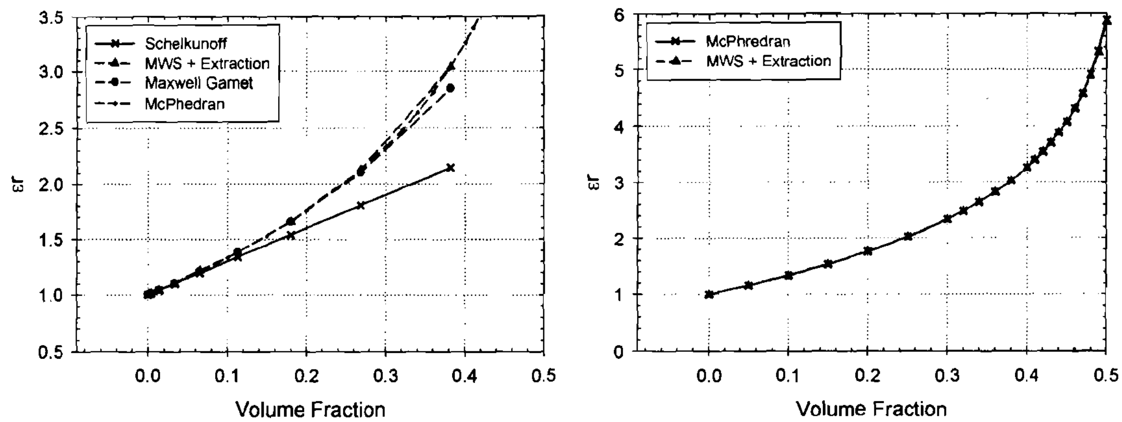


Figure 1.2 Extracted dielectric constant for a simple cubic lattice of perfectly conducting metallic sphere as the volume fraction changes compared with published results.

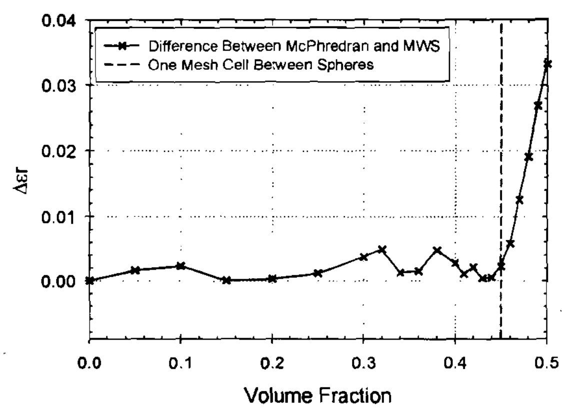


Figure 1.3 Difference between our extracted dielectric constant and McPhredran's published results for a simple cubic lattice of perfectly conducting metallic spheres as the volume fraction increases.

MG gives very good results up to 30% volume fraction or better.

This lattice is also diamagnetic. The effective μ_r is shown next, where the MG formula for μ_r is

$$\mu_{r,eff} = \frac{1-f_a}{1+\frac{f_a}{2}} \qquad (3)$$

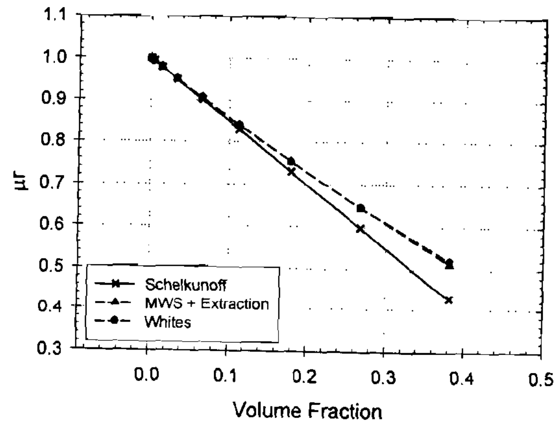


Figure 1.4 Extracted relative permeability for a simple cubic lattice of perfectly conducting metallic sphere as the volume fraction changes.

BCC lattice of PEC spheres: 20 unit cells for plane wave / plane slab MWS simulations. NRW to extract $\epsilon_{r,eff}$; $\mu_{r,eff}$ from S parameters.

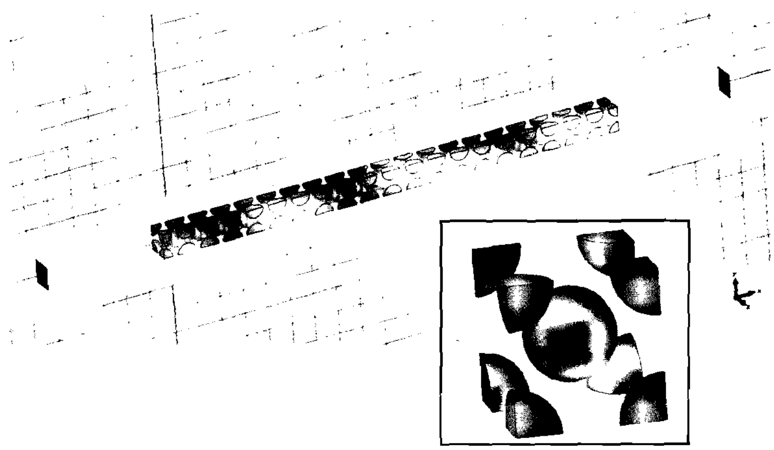


Figure 1.5 Simulation setup and unit cell of a body centered cubic lattice of perfectly conducting metallic spheres.

Published ϵ_r results below for [6] are from

McKenzie, McPhedran, and Derrick, Proc. Royal Soc. Lond., Vol. A. 362, 1978.

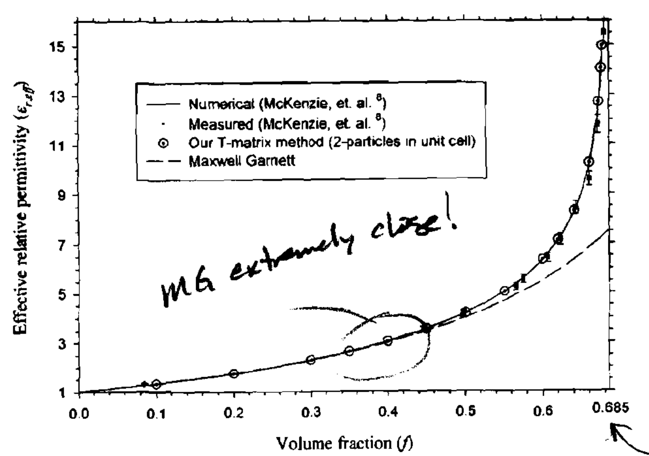
Table 1.1 Extracted and published body centered cubic lattice material parameters.

Volume Fraction	Published ε_r [6]	Extracted ε_r	Analytical μ_r	Extracted μ_r	ε_r Difference
0.000	1.000	1.000	1.000	1.000	0.000
0.050	1.158	1.158	0.927	0.921	0.000
0.100	1.333	1.331	0.857	0.853	0.002
0.150	1.530	1.532	0.791	0.783	0.002
0.200	1.751	1.753	0.727	0.722	0.002
0.250	2.002	2.001	0.667	0.660	0.001
0.300	2.292	2.292	0.609	0.603	0.000
0.350	2.631	2.633	0.553	0.544	0.002
0.400	3.035	3.037	0.500	0.491	0.002
0.450	3.532	3.531	0.449	0.440	0.001
0.500	4.166	4.168	0.400	0.389	0.002

McKenzie MWS + HWR MG MWS + HWR

Similarly good results w/ BCC lattice. Alternatively, here's a different method of computation for a BCC lattice of PEC spheres. Using a T matrix method:

Keith W. Whites



Percolation limit. Percolation for MG = 100% Not realistic.

FIG. 3. Computed $\varepsilon_{r,\mathrm{eff}}$ for a body-centered-cubic lattice of conducting spheres. The Maxwell Garnett results are from Eq. (29) and the T-matrix results are from Eq. (25) using the solution method described in Sec. II.

As recorded in that paper by Whites in Fig 6, MG accurate to 5% for BCC lattice up to lattices w/ 36% volume fraction. Nice!

MG formula can be used for other particle shapes as well, though with mixed results for relatively high volume fraction.

Cube particles:

Surprisingly, though, is the incredible accuracy of the MG formula for cube particles (or square in 2-D). Here showing results for a S.C. lattice of PEC cube particles:

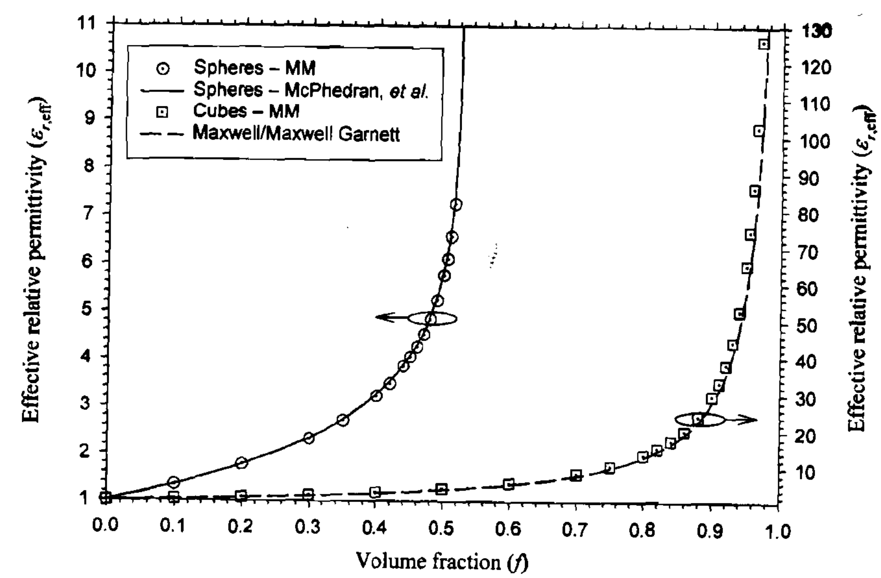


Fig. 3. Computed $\varepsilon_{r,\mathrm{eff}}$ for 3-D lattices of conducting spheres and conducting cubes. Our MM solution for conducting spheres is compared with data from [10]. Note that the vertical $\varepsilon_{r,\mathrm{eff}}$ scales for the sphere and cube data are much different.

WHITES AND WU: EFFECTS OF PARTICLE SHAPE ON EFFECTIVE PERMITTIVITY OF COMPOSITE MATERIALS

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MG almost exactly computes $\varepsilon_{r,\mathrm{eff}}$ for all volume fraction from 0 to 1! Amazing!

Experimental results undertaken in that same paper to verify that phenomenon. (measuring quasistatic σ_{eff}.)

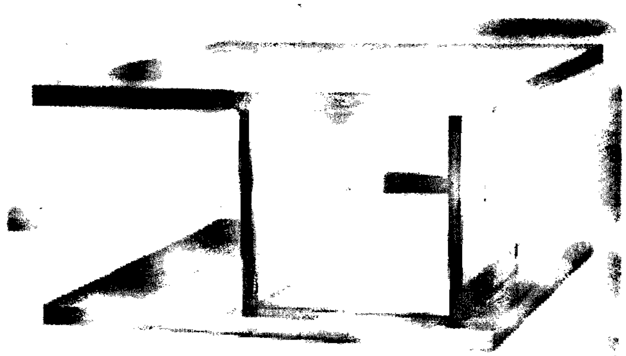

Fig. 6. Photograph of the conducting cube lattice apparatus for measuring quasi-static effective conductivity. The upper and lower plates as well as the cube are brass. The four-sided box is Lexan. For sample #2 shown above, the measured volume fraction was $f = 0.1952 \pm 0.0005$, as listed in Table I.

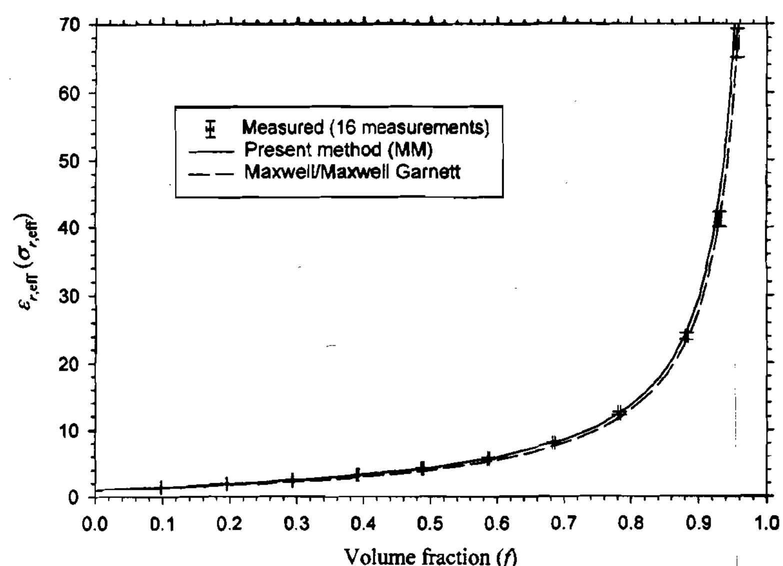

Fig. 8. Predicted quasi-static effective permittivity and measured effective conductivity for a simple cubic lattice of conducting cubes. The measurements were performed at 80 kHz. Error bars indicate one standard deviation in both $\sigma_{r,\mathrm{eff}}$ and volume fraction.

Because of this remarkable behavior of what we'll call cube media, we've been able to "engineer", or control/design, the effective material properties of different systems.

One example of this is changing the effective sheet impedance of resistive films (Kapton XC) by creating square shaped perforations of various sizes, but in a regular pattern. Can even grade the properties across the film.

(*) See the Metamaterials 2009 presentation by Whites and Glover. (*)