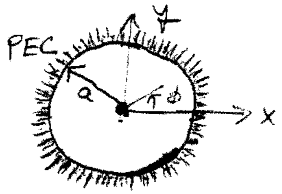
TE^z and TM^z modes in a Circular Waveguide.

We'll use the results from the previous lecture to determine the TE^z & TM^z modes that can propagate in a hollow, circular metallic waveguide. Wave propagation in $+z$ direction is assumed ($e^{-j\beta_z z}$).



TE^z Modes.

For TE^z modes, $E_z = 0$ & $H_z \neq 0$. From our discussions in the previous lecture, we expect standing waves in ρ direction, so choose the standing wave-type Bessel & Neumann fcts.

$$H_z = \underbrace{\left[A'\sin(n\phi) + B'\cos(n\phi)\right]\cdot\left[C'J_n(\beta_c\rho) + D'Y_n(\beta_c\rho)\right]}e^{-j\beta_z z} \quad (1)$$

for waves propagating in the $+z$ direction. Note that $\beta_c^2 = \beta_\rho^2 = \beta^2 - \beta_z^2$.

Also Notice that as $\rho \to 0$, $Y_n(\beta_c\rho) \to -\infty$. There is no reason to suspect the field to be singular along the axis $\rho = 0$, hence D' must be zero. Eqn (1) then becomes

$$H_z = \left(A\sin n\phi + B\cos n\phi\right)J_n(\beta_c\rho)\,e^{-j\beta_z z} \quad (2)$$

The boundary condition on the PEC outer wall at $\rho = a$ is $\hat{n}\times E = 0$. The only tangential E we need to consider is E_ϕ since $E_z = 0$ for TE^z modes. With $E_z = 0$, applying (2) from the previous lecture,

$$E_\phi = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial\rho} \underset{(2)}{=} \frac{j\omega\mu}{\beta_c^2}\left(A\sin n\phi + B\cos n\phi\right)\frac{\partial J_n(\beta_c\rho)}{\partial\rho}\,e^{-j\beta_z z} \quad (3)$$

Rather than keeping the derivative $\dfrac{\partial J_n(\beta_c\rho)}{\partial\rho}$, it is customary to cast this factor in terms of a derivative with respect to the argument of J_n. That is

$$\frac{\partial J_n(\beta_c\rho)}{\partial\rho} = \frac{\beta_c}{\beta_c}\frac{\partial J_n(\beta_c\rho)}{\partial\rho} = \beta_c \frac{\partial J_n(\beta_c\rho)}{\partial(\beta_c\rho)} = \beta_c J_n'(\beta_c\rho) \quad (4)$$

The prime notation indicates a derivative of J_n with respect to its argument. Consequently, (3) becomes

$$E_\phi = j\omega\mu \frac{\beta_c}{\beta_c^2}\left(A \sin n\phi + B\cos n\phi\right) J_n'(\beta_c\rho)\, e^{-j\beta_z z} \quad (5)$$

Applying the b.c. that $E_\phi(\rho=a) = 0 \quad \forall \phi, z$ to (5) requires that

$$J_n'(\beta_c a) = 0 \quad (6)$$

So, the first question is: does $J_n'(x) = 0$ for some x? The answer is yes. The second is: how many zeros of J_n' are there? The answer is an infinite number.

To calculate the derivative of a Bessel function, we can use recurrence formulas such as (Abramowitz and Stegun, 9.1.27):

$$B_{n-1}(x) - B_{n+1}(x) = 2B_n'(x) \quad (7)$$

or

$$B_n'(x) = -B_{n+1}(x) + \frac{n}{x}B_n(x) \quad (8)$$

where B_n denotes J_n, Y_n, $H_n^{(1)}$, $H_n^{(2)}$, or any linear combination of these fcts., and the prime indicates differentiation wrt the argument.

For example, using (8) we find that $J_0'(x) = -J_1(x)$, which is shown plotted on the next page. What we see is there are indeed zeros of J_0', and a very large number of them.

Defining the m^{th} $(m=1,2,3,\ldots)$ root of J_n' as p'_{nm} such that $J_n'(p'_{nm}) = 0$, then (6) is satisfied when

$$\beta_c a = p'_{nm} \quad \text{or} \quad \underline{\beta_{c,nm} = \frac{p'_{nm}}{a}} \qquad \begin{matrix} n = 0, 1, 2, \ldots \\ m = 1, 2, 3, \ldots \end{matrix} \qquad (9)$$

Values of p'_{nm} are tabulated in Abramowitz & Stegen, and elsewhere, or found numerically. From Pozar:

TABLE 3.3 Values of p'_{nm} for TE Modes of a Circular Waveguide

n	p'_{n1}	p'_{n2}	p'_{n3}
0	3.832	7.016	10.174
1	1.841 $= TE^z_{11}$	5.331	8.536
2	3.054	6.706	9.970

So which of these TE^z modes has the lowest cutoff freq.? To answer this question we start with the dispersion relation we derived in the last lecture $\beta_c^2 = \beta^2 - \beta_z^2$. In more specific terms,

$$\beta_{znm} = \sqrt{\beta^2 - \beta_{cnm}^2} \qquad \begin{matrix} n = 0, 1, 2, \ldots \\ m = 1, 2, 3, \ldots \end{matrix} \qquad (10)$$

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Needs["Graphics`Legend`"]
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In[49]:=
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Jnp[n_, x_] := -BesselJ[n + 1, x] + n / x * BesselJ[n, x]
Plot[{Jnp[0, x], BesselJ[0, x]}, {x, 0, 20},
 AxesLabel → {"x", None}, LegendShadow → None, LegendPosition → {0.1, 0},
 PlotLegend → {"J0'", "J0"}, PlotStyle → {Tomato, Dashing[{0.03}]}]
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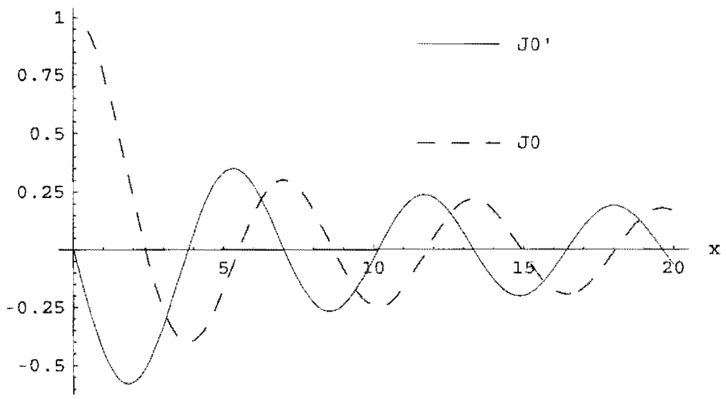


Out[50]= - Graphics -

Cutoff occurs when $\beta_{znm} = 0$, so from (10) we find the cutoff frequencies for TE^z_{nm} modes in a circular wvgd:

$$\omega_{c,nm}^2 \mu\epsilon = \beta_{c,nm}^2 \Rightarrow f_{c,nm} = \frac{\beta_{c,nm}}{2\pi\sqrt{\mu\epsilon}}$$

and using (9):

$$\underline{f_{c,nm} = \frac{p'_{nm}}{2\pi a\sqrt{\mu\epsilon}}} \qquad \begin{matrix} n = 0, 1, 2, \cdots \\ m = 1, 2, 3, \cdots \end{matrix} \qquad (11)$$

So to answer the question, we see from (11) & the values in Table 3.3 that the TE^z_{11} mode has the lowest cutoff freq. of the TE^z modes.

As shown in Pozar, 3rd ed, p.120, the TE modal impedance in circular wvgd is

$$Z_{TE} = \frac{E_\rho}{H_\phi} = \frac{-E_\phi}{H_\rho} = \frac{\omega\mu}{\beta_z} \qquad (12)$$

which is the same expression as TE^z modes in rectangular waveguides!

TM^z Modes in Circular Waveguide

For TM^z modes, $H_z = 0$ while $E_z \neq 0$. From our discussions in the previous lecture

$$E_z = \left[A'\sin(n\phi) + B'\cos(n\phi)\right] \cdot \left[C' J_n(\beta_c\rho) + D' Y_n(\beta_c\rho)\right] e^{-j\beta_z z} \qquad (13)$$

or sometimes written as $E_z = J_n(\,)\begin{Bmatrix}\sin(n\phi)\\ \cos(n\phi)\end{Bmatrix} e^{-j\beta_z z}$

As $\rho \to 0$, $Y_n(\beta_c\rho) \to -\infty$ which will require us to set $D' = 0$ so that

$$E_z = (A \sin n\phi + B \cos n\phi)\, J_n(\beta_c \rho)\, e^{-j\beta_z z} \qquad (14)$$

To satisfy the boundary condition $E_z(\rho=a) = 0 \;\; \forall \phi \;\&\; z$, then $J_n(\beta_c a) = 0$. This is satisfied when

$$\beta_c a = p_{nm} \quad \text{or} \quad \beta_{cnm} = \frac{p_{nm}}{a} \qquad (15)$$

$n = 0, 1, 2, \cdots$
$m = 1, 2, 3, \cdots$

where p_{nm} is the m^{th} zero of the Bessel fct of order n. A few of these zeros are listed below (Ref, Pozar, 3rd ed.):

Notice 3.832 for TM_{11} & 3.832 for TE_{01} modes. Why? $J_0'(\beta_c a) = -J_1(\beta_c a)$ root satisfies both eqns. (Generalize to all TE_{0m} modes for all m.)

TABLE 3.4 Values of p_{nm} for TM Modes of a Circular Waveguide

n	p_{n1}	p_{n2}	p_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

same values as $n=0$ row for TE^z modes.

Eqn. (15) is also sufficient to enforce $E_\phi(\rho=a) = 0 \;\; \forall \phi, z$. The cutoff frequencies of these TM^z modes occurs when $\beta_z = 0$, which means that

$$f_{c,nm} = \frac{\beta_{c,nm}}{2\pi\sqrt{\mu\epsilon}} \underset{(15)}{=} \frac{p_{nm}}{2\pi a\sqrt{\mu\epsilon}} \qquad n = 0, 1, 2, \cdots \quad m = 1, 2, 3, \cdots \qquad (16)$$

Using values for p_{nm} in (16) from the table above indicates that the TM_{01} mode has the lowest cutoff freq. among the TM^z modes. (What is the physical consequence of the "0" in the TM_{01} mode? No field variation in ϕ

As shown in Pozar (3rd ed., p. 122), the modal impedance for TM^z in circular waveguides is

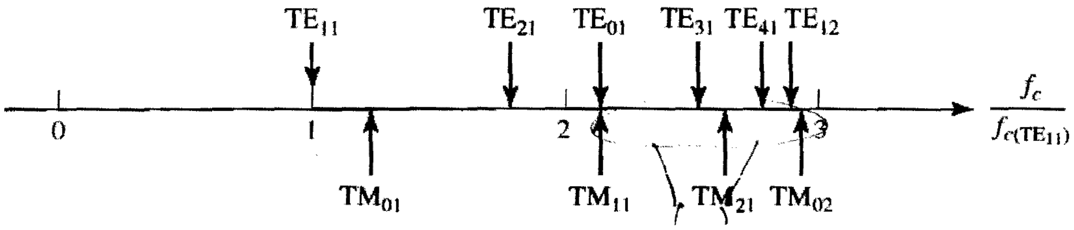
$$Z_{TM} = \frac{E_\rho}{H_\phi} = \frac{-E_\phi}{H_\rho} = \frac{\beta_z}{\omega\epsilon} \tag{17}$$

This is the same expression as for TM^z modes in rectangular waveguides.

Dominant Mode

The mode with the smallest cutoff frequency is the TE_{11} mode in the circular waveguide. This is followed by the TM_{01}, TE_{21}, TE_{01} / TM_{11} (degenerate modes), etc.

Relative to the cutoff frequency of the TE_{11} mode, the cutoff frequencies of the other modes in the circular waveguide are (Pozar, 3rd ed.):



Field Plots

Field plots of a few lowest order modes in the circular waveguide are shown on the next page.

A physical interpretation of the n and m indices can be observed in these field patterns. The index m is the number of periodic variations in the field pattern in ϕ, while n is the number of variations ~~of~~ the field pattern in ρ beginning from the center of the guide out to the wall.

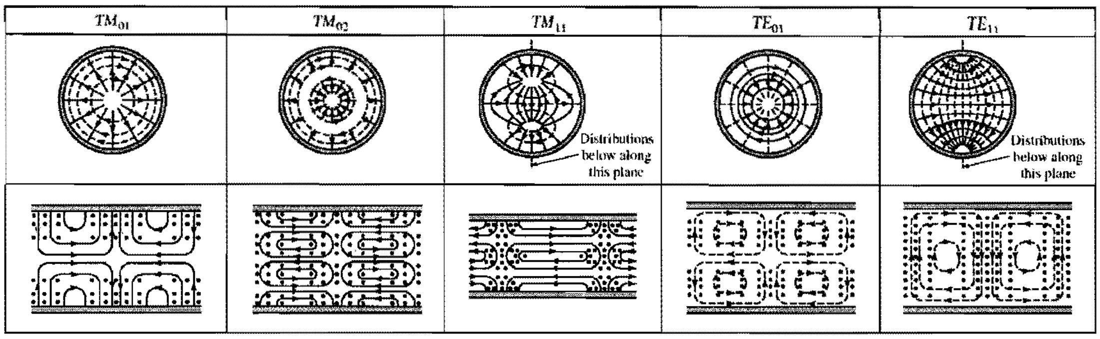


Figure 3.14 (p. 125)

Field lines for some of the lower order modes of a circular waveguide.

Reference: D. M. Pozar, *Microwave Engineering*. Hoboken, NJ: John Wiley & Sons, third ed., 2005.