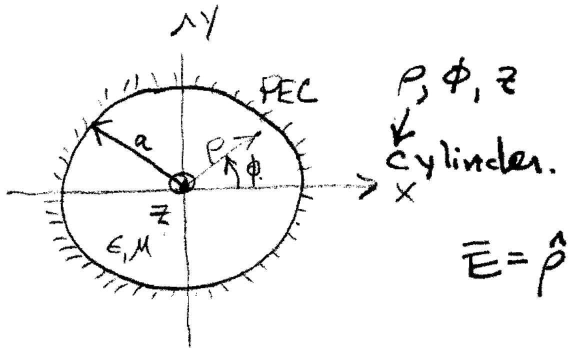
Circular Waveguides, Wave equation in cylindrical Coords. Properties of Bessel Functions

Another popular type of hollow metallic waveguide is one with a circular cross-section:



$$\bar{E} = \hat{\rho} E_\rho + \hat{\phi} E_\phi + \hat{z} E_z .$$

Because the interior of the waveguide is separable in the circular cylindrical coord. system, it will be necessary for us to solve the wave eqn. in that coord. system.

As with the rectangular wave guide, if we are searching for wave solutions prop in $\pm z$, it is possible to express the transverse fields that are fcts. of only the field comps E_z & H_z. For example, with prop in $+z$ as $e^{-j\beta_z z}$ (Pozar, 3rd, p. 118), then in a homogeneous space:

$\nabla \times \bar{E} = -j\omega\mu \bar{H}$ $\quad$ $\nabla \times \bar{H} = j\omega\epsilon \bar{E}$ $\Rightarrow$ expand circular cyl. coord. system.

$$E_\rho = \frac{-j}{\beta_c^2}\left(\beta_z \frac{\partial E_z}{\partial \rho} + \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right) \tag{1}$$

$$E_\phi = \frac{-j}{\beta_c^2}\left(\frac{\beta_z}{\rho} \frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial \rho}\right) \tag{2}$$

$$H_\rho = \frac{j}{\beta_c^2}\left(\frac{\omega\epsilon}{\rho} \frac{\partial E_z}{\partial \phi} - \beta_z\frac{\partial H_z}{\partial \rho}\right) \tag{3}$$

$$H_\phi = \frac{-j}{\beta_c^2}\left(\omega\epsilon \frac{\partial E_z}{\partial \rho} + \frac{\beta_z}{\rho}\frac{\partial H_z}{\partial \phi}\right) \tag{4}$$

$$\beta_c^2 \equiv \beta^2 - \beta_z^2$$

Once we have analytical solutions for E_z and H_z, we can determine all remaining field components. Consequently, it makes sense to separate the field solutions into TE^z & TM^z modes, as in a rectangular waveguide.

From our previous discussions, the wave eqn for time harmonic fields has the form

$$\nabla^2 \bar{T} + \beta^2 \bar{T} = 0 \tag{5}$$

where $\bar{T} = \bar{E}$ or $\bar{H}$. We're interested in finding solutions to this eqn in circular cylindrical coords where

$$\bar{T} = \hat{\rho}\, T_\rho(\rho,\phi,z) + \hat{\phi}\, T_\phi(\rho,\phi,z) + \hat{z}\, T_z(\rho,\phi,z) \tag{6}$$

Remember that $\nabla^2 \bar{T}$ is an operator defined on the rectangular comps of $\bar{T}$, so we can't just write (5) as three uncoupled eqns. for T_ρ, T_ϕ, & T_z.

Rather, we need to apply the definition of $\nabla^2 \bar{T}$:

$$\nabla \times (\nabla \times \bar{T}) = \nabla(\nabla \cdot \bar{T}) - \nabla^2 \bar{T}$$

or

$$\nabla^2 \bar{T} \equiv \nabla(\nabla \cdot \bar{T}) - \nabla \times (\nabla \times \bar{T}) \tag{7}$$

Sub. (7) into the wave eqn (5):

$$\nabla(\nabla \cdot \bar{T}) - \nabla \times (\nabla \times \bar{T}) + \beta^2 \bar{T} = 0 \tag{8}$$

Expanding out the L.H.S. of (8) and noting that in the circular cylindrical coord. system

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \qquad (9)$$

where $\Psi = \Psi(\rho, \phi, z)$ is a scalar fct of position, gives three second-order PDE's:

$$\hat{\rho}: \quad \nabla^2 T_\rho + \left(-\frac{T_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial T_\phi}{\partial \phi}\right) + \beta^2 T_\rho = 0 \qquad (10)$$

$$\hat{\phi}: \quad \nabla^2 T_\phi + \left(-\frac{T_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial T_\rho}{\partial \phi}\right) + \beta^2 T_\phi = 0 \qquad (11)$$

coupled.

$$\hat{z}: \quad \nabla^2 T_z + \beta^2 T_z = 0. \qquad (12)$$

Notice that (10) & (11) are coupled PDE's for T_ρ & T_ϕ. Tough to solve. However, (12) as same form for T_z that we've seen before! (Actually, not too big of a surprise since this is a cartesian direction as well). Plus, (1) – (4) allow us to compute all transverse fields once T_z is known.

So, that settles it: we'll solve (12) for T_z

Expanding the Laplacian operator in (12) in the circular cylindrical coord system as in (9) gives

$$\frac{\partial^2 T_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial T_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 T_z}{\partial \phi^2} + \frac{\partial^2 T_z}{\partial z^2} + \beta^2 T_z = 0 \qquad (13)$$

Following the separation of variables method, we'll search for sol'ns to T_z that have the product form:

$$T_z = f(\rho)\, g(\phi)\, h(z) \qquad (14)$$

Sub. (14) into (13) gives

$$gh \frac{\partial^2 f}{\partial^2 \rho^2} + \frac{gh}{\rho}\frac{\partial f}{\partial \rho} + \frac{fh}{\rho^2}\frac{\partial^2 g}{\partial \phi^2} + fg\frac{\partial^2 h}{\partial z^2} = -\beta^2 f g h.$$

Dividing by $T_z = fgh$ & noting all derivatives are total derivatives, then

$$\frac{1}{f}\frac{d^2 f}{d\rho^2} + \frac{1}{f\rho}\frac{df}{d\rho} + \frac{1}{g\rho^2}\frac{d^2 g}{d\phi^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = -\beta^2 \qquad (15)$$

The RHS of (15) is not a fct. of spatial coordinates, so the LHS cannot be either. Further, the last term on the LHS is not a fct of ρ & ϕ coords. We can argue that this term cannot be a fct. of z either if the LHS of (15) is to sum to a constant (i.e, a value independent of ρ, ϕ, z) $\forall\, \rho$ & ϕ. By this result,

$$\frac{1}{h}\frac{d^2 h}{dz^2} = -\beta_z^2 \qquad (16)$$

where $-\beta_z^2$ is a constant (i.e., not fct of ρ, ϕ, or z).

Sub. (16) into (15) and multiplying by ρ^2 gives

$$\frac{\rho^2}{f}\frac{d^2 f}{d\rho^2} + \frac{\rho}{f}\frac{df}{d\rho} + \frac{1}{g}\frac{d^2 g}{d\phi^2} + \rho^2(\beta^2 - \beta_z^2) = 0 \qquad (17)$$

We can observe that the $\frac{1}{g}\frac{d^2g}{d\phi^2}$ term is independent of the ρ & z coords. We can argue that this term is also not a fct of ϕ if the LHS of (17) is to sum to zero $\forall \rho$ & z. Consequently,

$$\frac{1}{g}\frac{d^2g}{d\phi^2} = -n^2 \qquad (18)$$

where $-n^2$ is a constant (possibly a complex number at this point in the derivation. Will show later that n must be an integer.)

Sub. (18) into (17)' and multiply by f gives

$$\rho^2\frac{d^2f}{d\rho^2} + \rho\frac{df}{d\rho} + \left[\rho^2(\beta^2-\beta_z^2) - n^2\right]f = 0 \qquad (19)$$

Defining $\beta_c^2 \equiv \beta^2 - \beta_z^2$ gives

$$\rho^2\frac{d^2f}{d\rho^2} + \rho\frac{df}{d\rho} + \left[(\beta_c\rho)^2 - n^2\right]f = 0 \qquad (20)$$

We have a second-order ordinary diff. eqn for $f(\rho)$. Can't reduce it any further. Solutions?

This ODE (20) is called Bessel's equation of order n. Solutions to (20) are in the form of infinite series, not unlike other tabulated fcts like trigonometric fcts.

move to "A"

What remains now is to solve for f, g & h since each has now been separated into a single differential eqn for each fct, as given in (16), (18), and (20).

- For h in (16).

$$\frac{d^2h}{dz^2} + \beta_z^2 h = 0 \qquad (21)$$

Solutions have the form

$$h(z) = C_1 e^{-j\beta_z z} + C_2 e^{+j\beta_z z} \qquad (22)$$

for propagating waves in ± z directions.

- For g in (18).

$$\frac{d^2g}{d\phi^2} + n^2 g = 0$$

Solutions have the form

$$g(\phi) = C_1 e^{-jn\phi} + C_2 e^{+jn\phi} \qquad (23)$$

The fact that a given pt (ρ, ϕ, z) must be the same point $(\rho, \phi + n2\pi, z)$ requires n to be an integer. Another way of stating this is all our EM field solutions must be single-valued fcts.

"A" For f in (17). This is a much more complicated ODE. It has no simple solutions.

Commonly used solutions to Bessel's equation are

$$f(\rho) = C_1 J_n(\beta_c \rho) + C_2 Y_n(\beta_c \rho) \qquad (24)$$

where J_n = Bessel fct., order n — For real argument, J_n & Y_n are real valued.
Y_n = Neumann fct., "

or

$$f(\rho) = C_1 H_n^{(1)}(\beta_c \rho) + C_2 H_n^{(2)}(\beta_c \rho) \qquad (25)$$

where $H_n^{(1)}$ = Hankel fct of 1st kind, order $n \equiv J_n + jY_n$
$H_n^{(2)}$ = Hankel fct. of 2nd kind, order $n \equiv J_n - jY_n$

For real arguments, $H_n^{(1)}$ & $H_n^{(2)}$ are complex valued.

The specific linear combinations used for $f(\rho)$, $g(\phi)$, and $h(z)$ depend a great deal on the type of problem that is being solved.

For example, in a cylindrical waveguide, propagating (or evanescent) modes would be adequately represented by an $e^{\mp j\beta_z z}$ dependence, tho a resonant cavity, would likely begin w/ $\cos(\beta_z z)$ and $\sin(\beta_z z)$. $\cos(n\phi)$ & $\sin(n\phi)$ commonly used in waveguide & resonant cavity solutions.

Plots of the various soln's to Bessel's eqn are shown on the following pages. Much insight into the behavior of these complicated fcts. can be found by considering their asymptotic forms:

$$\left.\begin{aligned} J_n(x) &\sim \sqrt{\frac{2}{\pi x}}\,\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right) \\ Y_n(x) &\sim \sqrt{\frac{2}{\pi x}}\,\sin\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right) \end{aligned}\right\} \text{ as } x \to \infty$$

These fcts are analogous to cos & sin fcts and are used to represent standing cylindrical waves.

Also,

$$\left.\begin{aligned} H_n^{(1)}(x) &\sim \sqrt{\frac{2}{\pi x}}\, e^{j\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)} \\ H_n^{(2)}(x) &\sim \sqrt{\frac{2}{\pi x}}\, e^{-j\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)} \end{aligned}\right\} \text{ as } x \to \infty$$

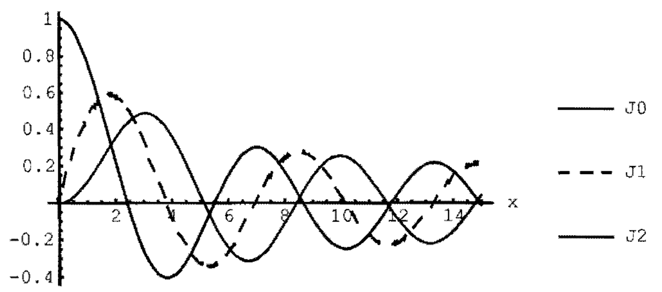
These fcts are analogous to $e^{jk_\rho \rho}$ & $e^{-jk_\rho \rho}$ and are used to represent cylindrical traveling wave behavior.

On the other extreme, notice that as $x \to 0$, only $J_n(x)$ does not become singular.

Plot Bessel and Hankel Functions

```
Needs["Graphics`Legend`"]

Plot[{BesselJ[0,x],BesselJ[1,x],BesselJ[2,x]},{x,0,15},
     AxesLabel -> {"x",None}, LegendShadow -> None,
     LegendPosition -> {1.1,-0.5},
     PlotLegend -> {"J0","J1","J2"},
     PlotStyle -> {GrayLevel[0],Dashing[{0.03}],
                   GrayLevel[0.5]}]
```

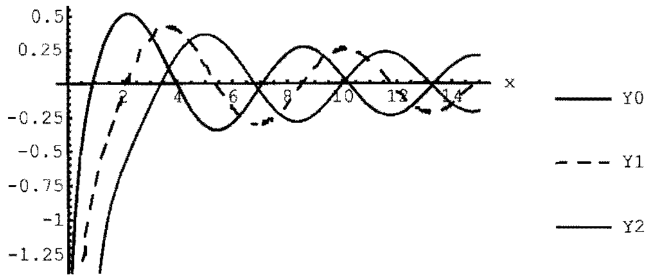


Standing wave behavior.

As $x \to 0$, only J_0 is non-zero. $J_0(0) = 1$.

- Graphics -

```
Plot[{BesselY[0,x],BesselY[1,x],BesselY[2,x]},{x,0,15},
     AxesLabel -> {"x",None}, LegendShadow -> None,
     LegendPosition -> {1.1,-0.5},
     PlotLegend -> {"Y0","Y1","Y2"},
     PlotStyle -> {GrayLevel[0],Dashing[{0.03}],
                   GrayLevel[0.5]}]
```

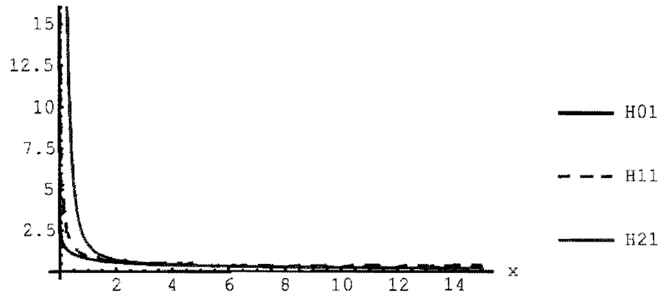


All are singular as $x \to 0$

- Graphics -

```
hankel1[n_,x_]:=BesselJ[n,x]+I*BesselY[n,x]
hankel2[n_,x_]:=BesselJ[n,x]-I*BesselY[n,x]
```

```
Plot[{Abs[hankel1[0,x]],Abs[hankel1[1,x]],
      Abs[hankel1[2,x]]},{x,0,15},
     AxesLabel -> {"x",None}, LegendShadow -> None,
     LegendPosition -> {1.1,-0.5},
     PlotLegend -> {"H01","H11","H21"},
     PlotStyle -> {GrayLevel[0],Dashing[{0.03}],
                   GrayLevel[0.5]}]
```

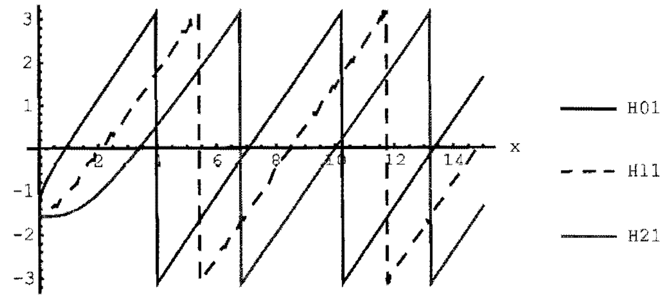


- $H_0^{(1)}$
- $H_1^{(1)}$
- $H_2^{(1)}$

$H_n^{(1)}(x)$ complex valued even for real argument.

- Graphics -

```
Plot[{Arg[hankel1[0,x]],Arg[hankel1[1,x]],
      Arg[hankel1[2,x]]},{x,0,15},
     AxesLabel -> {"x",None}, LegendShadow -> None,
     LegendPosition -> {1.1,-0.5},
     PlotLegend -> {"H01","H11","H21"},
     PlotStyle -> {GrayLevel[0],Dashing[{0.03}],
                   GrayLevel[0.5]}]
```

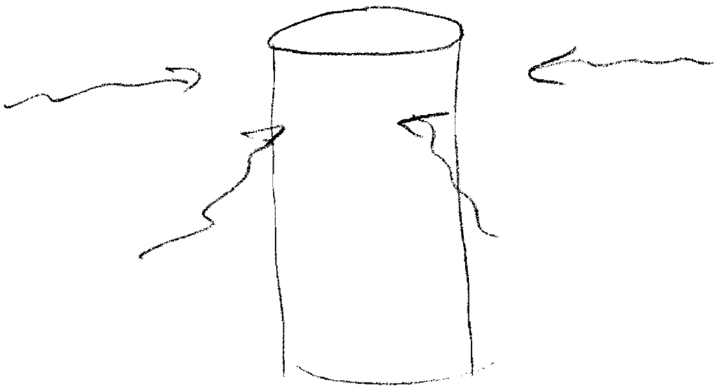


- $H_0^{(1)}$
- $H_1^{(1)}$
- $H_2^{(1)}$

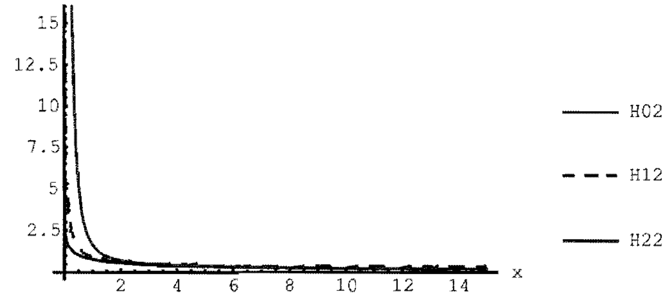
wave propagating behavior.

- Graphics -

inward propagating waves. Cylinder scattering



```
Plot[{Abs[hankel2[0,x]],Abs[hankel2[1,x]],
      Abs[hankel2[2,x]]},{x,0,15},
     AxesLabel -> {"x",None}, LegendShadow -> None,
     LegendPosition -> {1.1,-0.5},
     PlotLegend -> {"H02","H12","H22"},
     PlotStyle -> {GrayLevel[0],Dashing[{0.03}],
                   GrayLevel[0.5]}]
```



$- H_0^{(2)}$

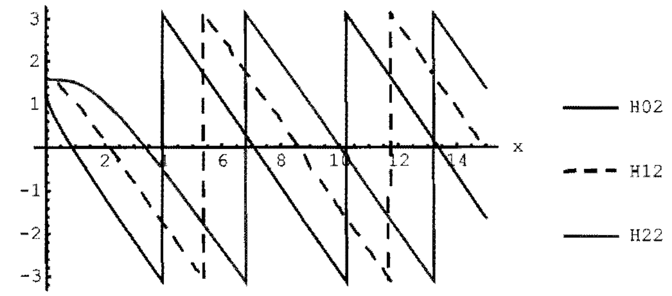
$- H_1^{(2)}$

$- H_2^{(2)}$

$H_n^{(2)}(x)$ Complex valued even for real argument

- Graphics -

```
Plot[{Arg[hankel2[0,x]],Arg[hankel2[1,x]],
      Arg[hankel2[2,x]]},{x,0,15},
     AxesLabel -> {"x",None}, LegendShadow -> None,
     LegendPosition -> {1.1,-0.5},
     PlotLegend -> {"H02","H12","H22"},
     PlotStyle -> {GrayLevel[0],Dashing[{0.03}],
                   GrayLevel[0.5]}]
```



$- H_0^{(2)}$

$- H_1^{(2)}$

$- H_2^{(2)}$

wave propagating behavior.

- Graphics -

outward propagating waves. cylinder scattering.