Material Measurement by Resonant Cavity Perturbation

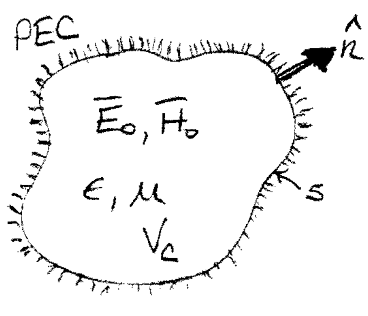
Resonant cavities can be an extremely accurate method for measuring the EM properties of low loss materials. In one approach the resonant frequencies and their associated Q's are measured with an empty cavity. Then a small sample is inserted into the cavity and the measurements are repeated for the down-ward shifted resonant frequencies and typically, the reduced Q's. (There are other methods for resonant cavity measurements that don't involve small specimens, but these won't be considered here.)

No calibration is typically required. Just measure changes in resonant f and changes to Q.

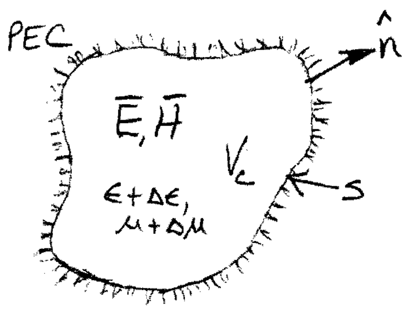
The difficult part is accurately extracting the EM properties from these resonant cavities. A <u>perturbation technique</u> is often employed for this purpose. This technique assumes that the sample only slightly changes, or perturbs, the fields that exist without the sample present.

The equations used to perform this extraction are derived beginning w/ the Maxwell curl equations for two situations:

<u>Unperturbed</u>



<u>Perturbed</u>



The perturbed case – for now – will assume the material <u>throughout</u> the cavity has been changed.

Exact Formula for Relative Change in Resonant Frequency

✓ Harrington, THEF pp. 318 - 323

✓ Pozar, 3rd 298-299

Then, in a source free region

$$-\nabla\times\bar{E}_o = j\omega_o\mu\bar{H}_o \qquad (1)$$

$$\nabla\times\bar{H}_o = j\omega_o\epsilon\bar{E}_o \qquad (2)$$

while,

$$-\nabla\times\bar{E} = j\omega(\mu+\Delta\mu)\bar{H} \qquad (3)$$

$$\nabla\times\bar{H} = j\omega(\epsilon+\Delta\epsilon)\bar{E} \qquad (4)$$

We've assumed the resonant frequencies of the cavity will change in the perturbed case, of course.

We scalar multiply (4) by $\bar{E}_o^*$ and the conjugate of (1) by $\bar{H}$:

$$\bar{E}_o^*\cdot\nabla\times\bar{H} = j\omega(\epsilon+\Delta\epsilon)\bar{E}_o^*\cdot\bar{E} \qquad (5)$$

$$-\bar{H}\cdot\nabla\times\bar{E}_o^* = -j\omega_o\mu^*\bar{H}\cdot\bar{H}_o^* \underset{\mu^*=\mu}{=} -j\omega_o\mu\bar{H}\cdot\bar{H}_o^* \qquad (6)$$

In (6) we've assumed the material has no magnetic loss so that $\mu^*=\mu$. Adding (5) and (6),

$$\bar{E}_o^*\cdot\nabla\times\bar{H} - \bar{H}\cdot\nabla\times\bar{E}_o^* = j\omega(\epsilon+\Delta\epsilon)\bar{E}_o^*\cdot\bar{E} - j\omega_o\mu\bar{H}\cdot\bar{H}_o^* \qquad (7)$$

Applying the vector i.d. $\nabla\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot\nabla\times\bar{A} - \bar{A}\cdot\nabla\times\bar{B}$ to the LHS of (7),

$$\nabla\cdot(\bar{H}\times\bar{E}_o^*) = j\omega(\epsilon+\Delta\epsilon)\bar{E}_o^*\cdot\bar{E} - j\omega_o\mu\bar{H}\cdot\bar{H}_o^* \qquad (8)$$

Likewise, we can perform analogous operations with (2) & (3) to find that

$$\nabla\cdot(\bar{H}_o^*\times\bar{E}) = j\omega(\mu+\Delta\mu)\bar{H}_o^*\cdot\bar{H} - j\omega_o\epsilon\bar{E}\cdot\bar{E}_o^* \qquad (9)$$

We've assumed $\epsilon^*=\epsilon$ (i.e., lossless dielectric material).

Next, we'll add (8) and (9) and integrate the sum throughout the volume of the cavity:

But $\Delta\epsilon$ & $\Delta\mu$ not assumed real!

$$\int_{V_c} \nabla\cdot\left(\bar{H}\times\bar{E}_o^* + \bar{H}_o^*\times\bar{E}\right) dv = \int_{V_c}\Big[j\omega(\epsilon+\Delta\epsilon)\,\bar{E}_o^*\cdot\bar{E} - j\omega_o\mu\,\bar{H}\cdot\bar{H}_o^* + j\omega(\mu+\Delta\mu)\,\bar{H}_o^*\cdot\bar{H} - j\omega_o\epsilon\,\bar{E}\cdot\bar{E}_o^*\Big] dv \tag{10}$$

Applying the divergence thm. to the LHS of (10) and combining terms on the RHS we find

$$\oint_S \left(\bar{H}\times\bar{E}_o^* + \bar{H}_o^*\times\bar{E}\right)\cdot \underbrace{d\bar{s}}_{\hat{n}\,ds} = j\int_{V_c}\Big\{\big[\omega(\epsilon+\Delta\epsilon)-\omega_o\epsilon\big]\bar{E}\cdot\bar{E}_o^* + \big[\omega(\mu+\Delta\mu)-\omega_o\mu\big]\bar{H}\cdot\bar{H}_o^*\Big\}\, dv \tag{11}$$

The integrand on the LHS of (11) is zero because

$$(\bar{H}\times\bar{E}_o^*)\cdot\hat{n} = \bar{H}\cdot(\bar{E}_o^*\times\hat{n}) = 0 \quad \text{on pec}$$

&

$$(\bar{H}_o^*\times\bar{E})\cdot\hat{n} = \bar{H}_o^*\cdot(\bar{E}\times\hat{n}) = 0 \quad \text{on pec.}$$

Consequently, (11) becomes

$$0 = \int_{V_c}\Big\{\big[\omega(\epsilon+\Delta\epsilon)-\omega_o\epsilon\big]\bar{E}\cdot\bar{E}_o^* + \big[\omega(\mu+\Delta\mu)-\omega_o\mu\big]\bar{H}\cdot\bar{H}_o^*\Big\}\, dv \tag{12}$$

s.t.

$$0 = \int_{V_c}(\omega\epsilon-\omega_o\epsilon)\,\bar{E}\cdot\bar{E}_o^*\, dv + \int_{V_c}(\omega\mu-\omega_o\mu)\,\bar{H}\cdot\bar{H}_o^*\, dv + \int_{V_c}\omega\Delta\epsilon\,\bar{E}\cdot\bar{E}_o^*\, dv + \int_{V_c}\omega\Delta\mu\,\bar{H}\cdot\bar{H}_o^*\, dv$$

or/

$$0 = (\omega-\omega_0)\int_{V_c}\left(\epsilon\,\bar{E}\cdot\bar{E}_0^* + \mu\,\bar{H}\cdot\bar{H}_0^*\right)dv$$
$$+\ \omega\int_{V_c}\left(\Delta\epsilon\,\bar{E}\cdot\bar{E}_0^* + \Delta\mu\,\bar{H}\cdot\bar{H}_0^*\right)dv$$

Consequently, we find that

$$\frac{\omega-\omega_0}{\omega} = \frac{-\int_{V_c}\left(\Delta\epsilon\,\bar{E}\cdot\bar{E}_0^* + \Delta\mu\,\bar{H}\cdot\bar{H}_0^*\right)dv}{\int_{V_c}\left(\epsilon\,\bar{E}\cdot\bar{E}_0^* + \mu\,\bar{H}\cdot\bar{H}_0^*\right)dv} \qquad (13)$$

This is an *exact* formula for the computation of the relative change in resonant frequency when a lossless cavity material is replaced with another (possibly lossy!) cavity material, assuming the cavity walls are pec. (Note that $\Delta\epsilon$ & $\Delta\mu$ were not assumed or required to be real.)

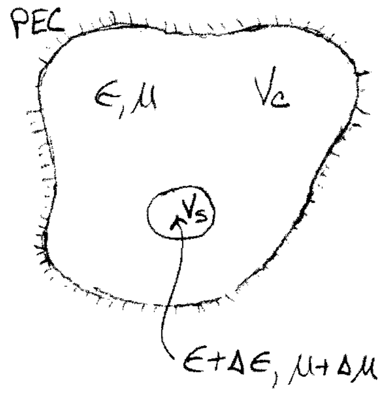
Perturbational Approximation

Eqn. (13) is an exact expression but it is difficult to use since $\bar{E}$ and $\bar{H}$ are generally not known. These are the fields inside the cavity w/ the material introduced.

In the case of a sample placed inside the cavity, (13) becomes no tell why?

$$\frac{\omega-\omega_0}{\omega} = \frac{-\int_{V_s}(\Delta\epsilon\,\bar{E}\cdot\bar{E}_0^* + \Delta\mu\,\bar{H}\cdot\bar{H}_0^*)\,dv}{\int_{V_c}(\epsilon\,\bar{E}\cdot\bar{E}_0^* + \mu\,\bar{H}\cdot\bar{H}_0^*)\,dv} \qquad (14)$$

where V_s is the volume of the specimen.



$V_s << V_c$

We will assume the specimen size is very small w.r.t. wavelength, and that the specimen volume is small w.r.t. V_c. In the denominator of (14) we'll approximate

$$\int_{V_c}(\epsilon\,\bar{E}\cdot\bar{E}_0^* + \mu\,\bar{H}\cdot\bar{H}_0^*)\,dv \approx \int_{V_c}\left((\epsilon|\bar{E}_0|^2) + \mu|\bar{H}_0|^2\right)dv \qquad (15)$$

This is reasonable because the contribution from V_s in the denominator of (14) is small wrt the remainder of the cavity volume for a small specimen perturbation.

(Not assuming $\Delta\epsilon$ and/or $\Delta\mu$ are small.)

(Check stored energy expression. Pozar has $\frac{1}{4}$ factor, rather than $\frac{1}{2}$.)

Notice that we're not necessarily assuming $\bar{E} \approx \bar{E}_o$ and $\bar{H} \approx \bar{H}_o$ within the specimen or in its vicinity. In fact, we haven't made any assumptions about $\Delta\epsilon$ or $\Delta\mu$. These may even be large. Rather, (15) is reasonable in the context that the contribution to the integral from V_s is small wrt the remainder of the cavity volume.

Consequently, (14) becomes

Harrington, eqn (7-10):

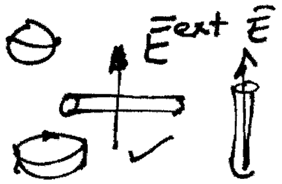
$$\frac{\omega - \omega_o}{\omega} \approx \frac{-\int_{V_s} (\Delta\epsilon\, \bar{E} \cdot \bar{E}_o^* + \Delta\mu \bar{H} \cdot \bar{H}_o^*)\, dv}{\int_{V_c} (\epsilon |E_o|^2 + \mu |H_o|^2)\, dv} \qquad (16)$$

There are quite a few paths we can take from here using (16). This equation is a common starting point for many methods to extract $\Delta\epsilon$ and/or $\Delta\mu$ of specimens in a resonant cavity.

Let's consider a nonmagnetic specimen ($\Delta\mu = 0$) or with a dielectric-magnetic specimen placed in a region of the cavity where $H_o = 0$, then (16) becomes

$$\frac{\omega - \omega_o}{\omega} \approx -\frac{\int_{V_s} \Delta\epsilon\, E \cdot E_o^*\, dv}{\int_{V_c} \epsilon |\bar{E}_o|^2\, dv}$$

$$= -\frac{\Delta\epsilon}{\epsilon} \frac{\int_{V_s} \bar{E} \cdot \bar{E}_o^*\, dv}{\int_{V_c} |E_o|^2\, dv} \qquad (17)$$



if we consider $C_1 = \dfrac{\int_{V_s} \bar{E} \cdot \bar{E}_o^* \, dv}{\int_{V_c} |E_o|^2 \, dv}$ (18) to be a factor relatively insensitive to sample permittivity, then we can use (17) to determine $\Delta\epsilon$.

To accomplish this, we use a specimen of known permittivity to calibrate by determining C_1 from the change in resonant frequency. Then specimens of similar shape and size can be determined from a change in resonant frequency using (17).

Another situation is small $\Delta\epsilon$ & $\Delta\mu$. In the limit as $\Delta\epsilon \to 0$ & $\Delta\mu \to 0$, there will only be a slight change to $\bar{E}$ and $\bar{H}$ from the unperturbed case. Hence, in (16) we can replace $\bar{E} \to \bar{E}_o$ & $\bar{H} \to \bar{H}_o$, giving

Harrington eqn (7-11):

$$\frac{\omega - \omega_o}{\omega} \approx \frac{-\int_{V_s} \left(\Delta\epsilon |\bar{E}_o|^2 + \Delta\mu |\bar{H}_o|^2\right) dv}{\int_{V_c} \left(\epsilon |\bar{E}_o|^2 + \mu |\bar{H}_o|^2\right) dv} \qquad (19)$$

From this result, if $\Delta\epsilon$ and/or $\Delta\mu$ increase the resonant frequency will <u>decrease</u>.

In the case of a non-magnetic ($\Delta\mu = 0$) or with the specimen placed in a region of the cavity where $\bar{H}_o \approx 0$, then (19) becomes

$$\frac{\omega-\omega_0}{\omega} \approx \frac{-\int_{V_s} \Delta\epsilon\,|E_0|^2 dv}{\int_{V_c} \epsilon |E_0|^2\, dv} = -\frac{\Delta\epsilon}{\epsilon} C_2 \qquad (20)$$

where

$$C_2 = \frac{-\int_{V_s} |E_0|^2 dv}{\int_{V_c} |E_0|^2 dv} \qquad (21)$$

Here, C_2 is only a function of the cavity geometry and the position of the specimen. This can be determined by calibration with a known specimen, but here the specimen & the calibration specimen don't necessarily need to be ~~the~~ same shape. Changes in resonant frequency in (20) can be used to find $\Delta\epsilon$.

Add discussion on complex ω. Use to determine ϵ' & ϵ'' from $\Delta\omega$ & ΔQ measurements. Use lossless equations & complex ω. $\left[\underset{\text{lossless resonator}}{\omega_0} \rightarrow \underset{\text{lossy resonator}}{\omega_0}\left(1+\frac{j}{2Q_0}\right) \right]$
Pozar. eqn (6.10)

Notice in (17) & (20) that ω will be a complex number is specimen permittivity is complex ($\Delta\epsilon$ = complex). The complex angular frequency is related to the real resonant frequency of the cavity & Q of the cavity by

$$\omega = \omega_r + j\omega_i$$

$$\omega_r = 2\pi f_c$$

$$\text{\&} \quad Q = \frac{\omega_r}{2\omega_i}$$