Resonant devices with extremely high Q can be realized by placing metal end caps onto a rectangular waveguide.

Energy is coupled into the cavity through a small hole (or hole) on a side wall, or by a small wire or loop extended through a side wall. We'll assume the excitation occurs on a \( z = \) constant side wall.

A wave excited at \( z = 0 \), for example, may prop on a waveguide mode in +z, reflect off the wall at \( z = d \), then prop back to the \( z = 0 \) wall.

At certain frequencies, these incident and reflected waves will add together in phase, strengthening the oscillations. Other than at these special frequencies, the waves interfere destructively and no energy can be coupled into the cavity.

We can determine these special frequencies by applying b.c.'s at \( z = 0 \) : \( z = d \).
We must be a bit careful here to apply the b.c.'s to the total \( \mathbf{E} \tan \) at these surfaces.

**TE cavity modes.** With propagation in the \( \pm z \) directions, then

For \( +z \) prop: \[
H_z^+ = A_{mn} \cos \alpha \cos \beta \ e^{-j \beta_z z}
\]

\[
E_x^+ = \frac{j \omega \mu \beta_{mn}}{\varepsilon_{\text{eff}}} A_{mn} \cos \alpha \sin \beta \ e^{-j \beta_z z}
\]

\[
E_y^+ = \frac{-j \omega \mu \beta_{mn}}{\varepsilon_{\text{eff}}} A_{mn} \sin \alpha \cos \beta \ e^{-j \beta_z z}
\]

For \( -z \) prop, in the \(-z\) direction, \( \beta_z \to -\beta_z \) in (1)-(3) gives

\[
H_z^- = B_{mn} \cos \alpha \cos \beta \ e^{+j \beta_z z}
\]

\[
E_x^- = \frac{j \omega \mu \beta_{mn}}{\varepsilon_{\text{eff}}} B_{mn} \cos \alpha \sin \beta \ e^{+j \beta_z z}
\]

\[
E_y^- = \frac{-j \omega \mu \beta_{mn}}{\varepsilon_{\text{eff}}} B_{mn} \sin \alpha \cos \beta \ e^{+j \beta_z z}
\]

Using (2), (5), the total \( E_x = E_x^+ + E_x^- \) \( \Rightarrow \)

\[
E_x^+ = \frac{j \omega \mu \beta_{mn}}{\varepsilon_{\text{eff}}} \cos \alpha \sin \beta 
\left( A_{mn} e^{-j \beta_z z} + B_{mn} e^{+j \beta_z z} \right)
\]

\( E_x^+ \) already satisfies b.c.'s at \( x=0, a \) and \( y=0, b \).
That's how we originally determined \( \beta_{xm} = \frac{\omega \mu y}{a} \) \( \beta_{yn} = \frac{\omega \mu x}{b} \).

Now, we need to apply boundary conditions at \( z=0 \) \( \Rightarrow \) for \( \mathbf{E} \tan = 0 \).
using (7). For \( E_x = 0 \),

\[
E_x = \frac{j \omega \mu \beta_{m n}}{\beta_{m n}} A_{m n} \sin^b \sin^a (\beta_{x} x + \beta_{y} y)
\]

so that

\[
\Rightarrow \quad \beta_{m n} = -A_{m n}
\]

Sub. (8) into (7):

\[
Ex = \frac{j \omega \mu \beta_{m n}}{\beta_{m n}} A_{m n} (e^{-j \beta_{z} z} - e^{+j \beta_{z} z})
\]

\[
= -j 2 \sin (\beta_{z} z)
\]

\[
E_x = \frac{2 \omega \mu \beta_{m n}}{\beta_{m n}} A_{m n} \cos^a \sin^b \sin (\beta_{z} z)
\]

At \( z = d \), \( E_x = 0 \)

\[
E_x = \frac{2 \omega \mu \beta_{m n}}{\beta_{m n}} A_{m n} \cos^a \sin^b \sin (\beta_{z} d)
\]

\[
\Rightarrow \quad \beta_{z} = \beta_{z p} = \frac{p \pi}{d}, \quad p = 1, 2, 3, \ldots
\]

It can be shown that this same condition (10) will enforce the b.c. \( E_y = 0 \) at \( z = 0, d \) since using (3) and (6)

\[
E_y = E_y^+ + E_y^- = \frac{2 \omega \mu \beta_{m n}}{\beta_{m n}} A_{m n} \sin^a \cos^b \sin (\beta_{z} z)
\]

With \( \beta_{z} = \beta_{z p} \) in (10), \( E_y \) in (11) vanishes at \( z = 0, d \).

Notice that sin (10), \( p \neq 0 \). If \( p = 0 \) in (9) or (11), then \( E_x = E_y = 0 \) \( \forall x, y, z \) inside the cavity. Because \( E_x = 0 \) for the TE \(_{m n}^p \) modes, then \( E = 0 \) if \( p = 0 \). With \( E = 0 \), no resonance is possible. (Can also show that \( H_x = 0 \) for
\[ \rho = 0 \Rightarrow \text{all } \text{TE}^2 \text{ fields vanish.} \]

Substituting (10) into the dispersion relation

\[ \beta_{xm}^2 + \beta_{yn}^2 + \beta_z^2 = \beta^2 = \omega^2 \mu \epsilon \]

then

\[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 + \left( \frac{p \pi}{d} \right)^2 = \omega^2 \mu \epsilon \]

(12)

The only degree of freedom left in the frequency of operation. From (12), the resonant frequencies of the cavity are thus

\[ f_{mp} = \frac{1}{2 \pi \sqrt{\mu \epsilon}} \sqrt{\left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 + \left( \frac{p \pi}{d} \right)^2} \]

(13)

for \( \text{TE}^2_{mp} \) modes, where \( m, n = 0, 1, 2, \ldots \), \( m = n \neq 0 \), \( p = 1, 2, 3, \ldots \)

\( \bullet \)

\( \text{TM}^2 \) cavity modes. Referencing Pozar, 3rd edition, Ch. 3, eqns. (3.100) and (3.101), for propagation in the \(+z\) direction:

\[ E_z^+ = A mn \sin x \sin y e^{-j \beta_{zm} z} \]

(14)

and

\[ E_x^+ = \frac{j \beta_z}{\beta_{zm}^2} \frac{\partial E_z^+}{\partial x} = \frac{j \beta_{zm}}{\beta_{zm}^2} A mn \beta_{xm} \cos x \sin y e^{-j \beta_{zm} z} \]

(15)

\[ E_y^+ = \frac{j \beta_z}{\beta_{zm}^2} \frac{\partial E_z^+}{\partial y} = \frac{j \beta_{zm}}{\beta_{zm}^2} A mn \beta_{ym} \sin x \cos y e^{-j \beta_{zm} z} \]

(16)

For propagation in the \(-z\) direction \( (\beta_{zm} \rightarrow -\beta_{zm}) \) then
\[ E_{x}^{-} = B_{mn} \sin^{2} \alpha \sin^{2} \beta e^{+i\beta_{x} z} \]  
\[ \text{(17)} \]

and

\[ E_{x}^{-} = \frac{+iB_{z} \beta_{mn}}{B_{nn}} B_{mn} \cos^{2} \beta \sin^{2} \beta e^{+i\beta_{z} z} \]  
\[ \text{(18)} \]

\[ E_{y}^{-} = \frac{+iB_{z} \beta_{mn}}{B_{nn}} B_{mn} \sin^{2} \alpha \cos^{2} \beta e^{+i\beta_{z} z} \]  
\[ \text{(19)} \]

Comparing (15), (16), (18), and (19) with the transverse \( E \) for \( TE_{2}^{m} \) modes in (2), (3), (5), and (6), we see that the transverse \( E \) for \( TE_{2}^{m} \) modes have the same spatial dependencies. Consequently, applying boundary conditions at \( z = 0 \) and \( d \), we will arrive at the same resonant frequency expression as in (13).

It turns out, though, that the index \( p = 0 \) is allowed for \( TM_{2}^{m} \) modes. How can this be since \( E_{y} = E_{x} = 0 \) \( \forall x, y, z \) when \( p = 0 \)? For \( TM_{2}^{m} \) modes, \( E_{y} \neq 0 \). From (14) and (17) with \( B_{mn} = A_{mn} \):

\[ E_{x}^{+} = E_{x}^{-} + E_{x}^{-} = A_{mn} \sin^{2} \alpha \sin^{2} \beta \left( e^{+i\beta_{x} z} + e^{+i\beta_{y} z} \right) \]

\[ = 2 A_{mn} \sin^{2} \alpha \sin^{2} \beta \cos (\beta_{z} z) \]  
\[ \text{(20)} \]

For \( p = 0 \), \( E_{x}^{+} = 2 A_{mn} \sin^{2} \alpha \sin^{2} \beta \neq 0 \).

So, for \( TM_{2}^{m} \) modes with \( p = 0 \), \( E_{x}^{+} \) is constant in \( z \) but varies in \( x, y \).

What about \( H \)? Can show that \( H_{x}, H_{y} \propto \cos \left( \frac{\beta_{x} z}{d} \right) \).
Such that \( H_x, H_y \neq 0 \). Consequently, \( E_z, H_x, \) and \( H_y \) are non-zero so that \( TM_{mnp}^2 \) modes are possible.

Consequently, for \( TM_{mnp}^2 \) modes, the resonant frequencies are

\[
F_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2}
\]  

(21)

where

\[
m, n = 1, 2, 3, \ldots \quad \text{and} \quad p = 0, 1, 2, \ldots
\]

---

Dominant Mode in a Rectangular Cavity Resonator

For the case when the cavity dimensions are related by \( b < a < c \), as sketched on p. 1, (13) and (21) to determine that the cavity mode with the lowest resonant frequency is \( TE_{101} \). This is the dominant or fundamental cavity mode.

The \( TE_{101} \) mode corresponds to a \( TE_{10} \) waveguide mode in a \( \frac{2\pi}{a} \)-long cavity.

Which mode resonates next as the frequency increases depends on the specific ratios of \( \frac{b}{a} \) and \( \frac{c}{a} \).
For example, in the case of a WR-90 waveguide that is 2 cm long, in $Z$: 

<table>
<thead>
<tr>
<th>Resonant frequency (GHz)</th>
<th>mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.959</td>
<td>$TE_{101}$ ✓</td>
</tr>
<tr>
<td>15.105</td>
<td>$TE_{201}$ ✓</td>
</tr>
<tr>
<td>16.146</td>
<td>$TM_{110}$ ✓</td>
</tr>
<tr>
<td>16.362</td>
<td>$TE_{02}$ ✓</td>
</tr>
<tr>
<td>16.549</td>
<td>$TE_{011}$ ✓</td>
</tr>
<tr>
<td>17.800</td>
<td>$TE_{111}$, $TM_{111}$ ✓</td>
</tr>
<tr>
<td>19.740</td>
<td>$TM_{210}$ ✓</td>
</tr>
<tr>
<td>19.917</td>
<td>$TE_{202}$ ✓</td>
</tr>
</tbody>
</table>

These are the resonant frequencies through 20 GHz. Notice:

- As the frequency increases, the "density" of resonant modes increases.
- At 17.800 GHz, both $TE_{111}$ & $TM_{111}$ can resonate.

Two modes that have the same resonant frequency but different field patterns are called degenerate modes.