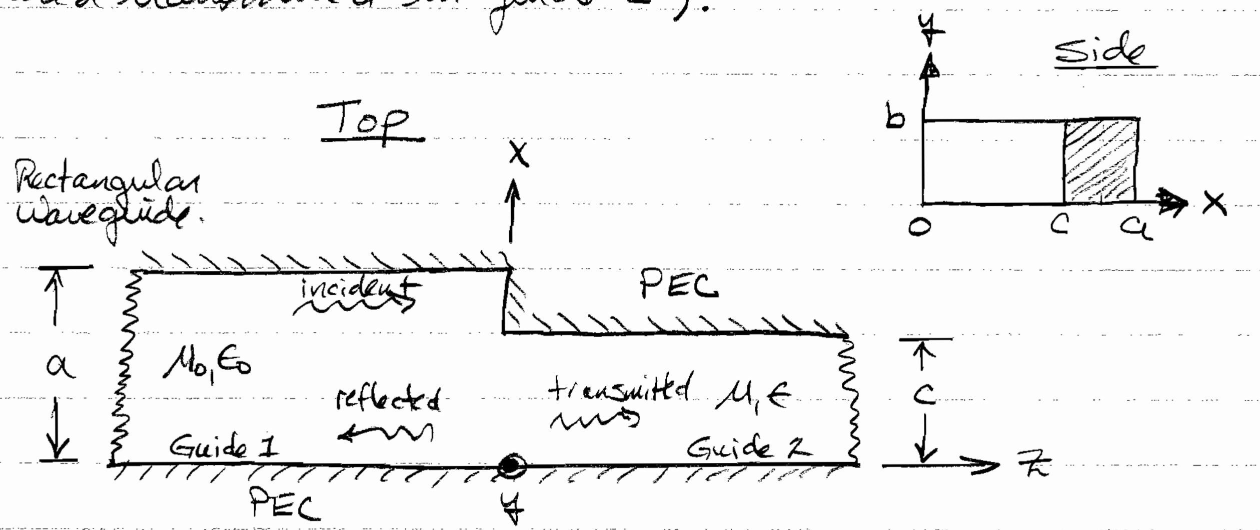
H-Plane Step Discontinuity in a Rectangular Waveguide

We learned in the previous lecture that for the H-plane discontinuity geometry shown below, an incident TE_{10} mode will excite only TE_{m0} modes (reflected in guide 1 and transmitted in guide 2).



The incident wave is a TE_{10} mode:

$$E_y^i = \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_{z,10}^a z} \tag{1}$$

$$H_x^i = \frac{-E_y^i}{Z_{TE_{10}}} = \frac{-\beta_{z,10}^a}{\omega\mu_0} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_{z,10}^a z} \tag{2}$$

$$E_x^i = H_y^i = 0$$

Referencing the previous lecture, the reflected waves are TE_{m0} modes where

$$E_y^r = \sum_{m=1}^{\infty} A_m \sin_x^a e^{+j\beta_{zm}^a z} \tag{3}$$

$$H_x^r = \sum_{m=1}^{\infty} \frac{\beta_{zm}^a}{\omega\mu_0} A_m \sin_x^a e^{+j\beta_{zm}^a z} \tag{4}$$

$$E_x^r = H_y^r = 0$$

and the transmitted waves are TE_{p0} modes where

$$E_y^t = \sum_{p=1}^{\infty} B_p \sin_x^c e^{-j\beta_{zp}^c z} \qquad (5)$$

$$H_x^t = \sum_{p=1}^{\infty} \frac{-\beta_{zp}^c}{\omega\mu} B_p \sin_x^c e^{-j\beta_{zp}^c z} \qquad (6)$$

$$E_x^t = H_y^t = 0.$$

(Pozer uses same index for reflected & transmitted modes.)

In the mode matching solution, our goal is to determine the A_m & B_p coeffs that enforce the field boundary conditions that have introduced by the step discontinuity.

- Tangential $\bar{E}$ continuous: At $z=0$ and for $0 \le y \le b$,

— $E_{y1} = E_{y2}$ for $0 \le x \le c$

— $E_{y1} = 0$ for $c \le x \le a$.

$E_{y_{1,2}}$ are the total fields in regions 1 & 2, respectively, so that this b.c. eqn. reads

$$E_y^i + E_y^r = \begin{cases} E_y^t & 0 \le x \le c \\ 0 & c \le x \le a \end{cases} \qquad (7)$$

Substituting (1), (3), and (5) into (7) gives

$$\sin\left(\frac{\pi x}{a}\right) + \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) = \begin{cases} \sum_{p=1}^{\infty} B_p \sin\left(\frac{p\pi x}{c}\right) & 0 \le x \le c \\ 0 & c \le x \le a \end{cases} \quad (8)$$

- <u>Tangential $\bar{H}$ continuous</u>: At $z=0$ and for $0 \le y \le b$:

$$H_{x_1} = H_{x_2} \qquad 0 \le x \le c. \quad (9)$$

There's no <u>constraint</u> on $\bar{H}_{tan}$ on pec. Sub. (2), (4), and (6) into (9) gives

$$-\frac{\beta^a_{z,10}}{\omega\mu_0} \sin\left(\frac{\pi x}{a}\right) + \sum_{m=1}^{\infty} \frac{\beta^a_{zm}}{\omega\mu_0} A_m \sin\left(\frac{m\pi x}{a}\right) = \sum_{p=1}^{\infty} -\frac{\beta^c_{zp}}{\omega\mu} B_p \sin\left(\frac{p\pi x}{c}\right) \quad (10)$$

Equations (8) and (9) need to be solved for the coeffs. A_m & B_p. These are a pair of doubly infinite, linear, constant coefficient equations. Their solution relies on the orthogonal properties of the trigonometric functions in (8) & (9).

In particular, we will take advantage of the orthogonality of sine functions over appropriate ranges:

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx = 0 \quad m \ne n \quad (11)$$

and

$$\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx = \frac{a}{2} \quad (12)$$

To accomplish this we multiply (8) by $\sin\left(\frac{m'\pi x}{a}\right)$ and integrate in x from 0 to a:

$$\int_0^a \sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)dx + \sum_{m=1}^{\infty} A_m \int_0^a \sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \sum_{p=1}^{\infty} B_p \int_0^c \sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{p\pi x}{c}\right)dx \qquad (13)$$

Using (11) and (12):

- $$\int_0^a \sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)dx = \begin{cases} \frac{a}{2} & m'=1 \\ 0 & \text{otherwise} \end{cases}$$

  Using the Kronecker delta symbol δ_{mn} ($=1$ if $m=n$, 0 otherwise) we can write this outcome more compactly as:

  $$\int_0^a \sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)dx = \frac{a}{2}\delta_{m'1} \qquad (14)$$

- $$\int_0^a \sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \frac{a}{2}\delta_{mm'} \qquad (15)$$

- $$\int_0^c \sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{p\pi x}{c}\right)dx = \frac{(-1)^p \frac{p\pi}{c}}{\left(\frac{m'\pi}{a}\right)^2-\left(\frac{p\pi}{c}\right)^2}\cdot\sin\left(m'\pi\frac{c}{a}\right) \equiv I_{m'p} \qquad (16)$$

Using (14)–(16) in (13) leaves

$$\frac{a}{2}\delta_{m'1} + A_{m'}\frac{a}{2} = \sum_{p=1}^{\infty} I_{m'p}B_p \qquad m'=1,2,3,\dots \qquad (17)$$

In a similar fashion, we'll solve for B_{p0} from (10) by multiplying this equation by $\sin\left(\frac{p'\pi x}{c}\right)$ and integrating over the range $x \in \{0, c\}$. This gives

$$-\frac{\beta^a_{z,10}}{\omega\mu_0}\int_0^c \sin\left(\frac{p'\pi x}{c}\right)\sin\left(\frac{\pi x}{a}\right)dx + \sum_{m=1}^{\infty}\frac{\beta^a_{zm}}{\omega\mu_0}A_m\int_0^c \sin\left(\frac{p'\pi x}{c}\right)\sin\left(\frac{m\pi x}{a}\right)dx$$
$$= \sum_{p=1}^{\infty}\frac{-\beta^c_{zp}}{\omega\mu}B_p\int_0^c \sin\left(\frac{p'\pi x}{c}\right)\sin\left(\frac{p\pi x}{c}\right)dx \qquad (18)$$

using (14)-(16), (18) becomes:

$$-\frac{\beta^a_{z,10}}{\omega\mu_0}I_{1p'} + \sum_{m=1}^{\infty}\frac{\beta^a_{zm}}{\omega\mu_0}A_m I_{mp'} = \frac{-\beta^c_{zp'}}{\omega\mu}B_{p'}\cdot\frac{c}{2} \qquad p'=1,2,3,\dots \qquad (19)$$

We'll substitute (19) into (17) to form an equation to solve for A_m. First, we'll replace $p' \to p$ in (19) to find

$$B_p = \left(\frac{-2\omega\mu}{c\,\beta^c_{zp}}\right)\left(\frac{-\beta^a_{z,10}}{\omega\mu_0}\right)I_{1p} + \left(\frac{-2\omega\mu}{c\,\beta^c_{zp}}\right)\sum_{m=1}^{\infty}\frac{\beta^a_{zm}}{\omega\mu_0}I_{mp}A_m \qquad p=1,2,3,\dots \qquad (20)$$

Substituting this into (17), with $m' \to m$, gives

$$\frac{a}{2}\delta_{m1} + \frac{a}{2}A_m = \sum_{p=1}^{\infty}I_{mp}\left[\frac{2\mu_r}{c}\frac{\beta^a_{z10}}{\beta^c_{zp}}I_{1p} - \frac{2\mu_r}{c\,\beta^c_{zp}}\sum_{k=1}^{\infty}\beta^a_{zk}I_{kp}A_k\right] \qquad m=1,2,3,\dots \qquad (21)$$

To avoid confusion, we relabeled the index in the last summation with k rather than m.

Eqn (21) can be rearranged as

$$\frac{a}{2}A_m + \frac{2\mu_r}{C}\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\frac{\beta^a_{zk}}{\beta^c_{zp}} I_{mp} I_{kp} A_k = \frac{2\mu_r}{c}\beta^a_{z10}\sum_{p=1}^{\infty}\frac{1}{\beta^c_{zp}} I_{mp} I_{1p} - \frac{a}{2}\delta_{m1} \quad m=1,2,3,\ldots \quad (22)$$

No analytical solution for the coeffs A_m. However, if we truncate the infinite summations in (22) by chosing a maximum of M scattered modes in guide 1 & a maximum of P scattered modes in guide 2, we can find a numerical solution for the coeffs A_m in (22).

To facilitate this numerical solution, it is usually extremely helpful to cast (22) in a matrix eqn. form:

$$\bar{\bar{Q}}\cdot\bar{A} = \bar{P} \quad (23)$$

where $\bar{\bar{Q}}$ is an $M\times M$ matrix with elements (exchanged p & k sums in 22):

$$Q_{ij} = \frac{a}{2}\delta_{ij} + \frac{2\mu_r}{C}\sum_{p=1}^{P}\frac{\beta^a_{zj}}{\beta^c_{zp}} I_{ip} I_{jp} \quad (24)$$

$i,j=1,2,\ldots,M$

Pozar has I_{pj}

$\bar{P}$ is the $M\times 1$ "excitation" vector:

Pozar has I_{p1}

$$\bar{P}_i = \frac{2\mu_r}{C}\beta^a_{z10}\sum_{p=1}^{P}\frac{1}{\beta^c_{zp}} I_{ip} I_{1p} - \frac{a}{2}\delta_{i1} \quad (25)$$

$i=1,2,\ldots,M$

and $\bar{A}$ is the vector of unknown TE_{i0} mode coefficients

$$A_i = A_{i0} \qquad i = 1, 2, \cdots, M. \qquad (26)$$

Can fairly easily code up (23) – (25) to solve for A_i coeffs. These coeffs can then subsequently be used in (20) to determine the B_{p0} mode coeffs in guide 2.

Ratio of $\frac{A_1}{I}$ is ref. coeff. for TE_{10}, while ratio $\frac{B_1}{I}$ is TE_{10} trans. coeff into guide 2.

Results from Pozar, 3rd ed., p. 203 shown on next page.

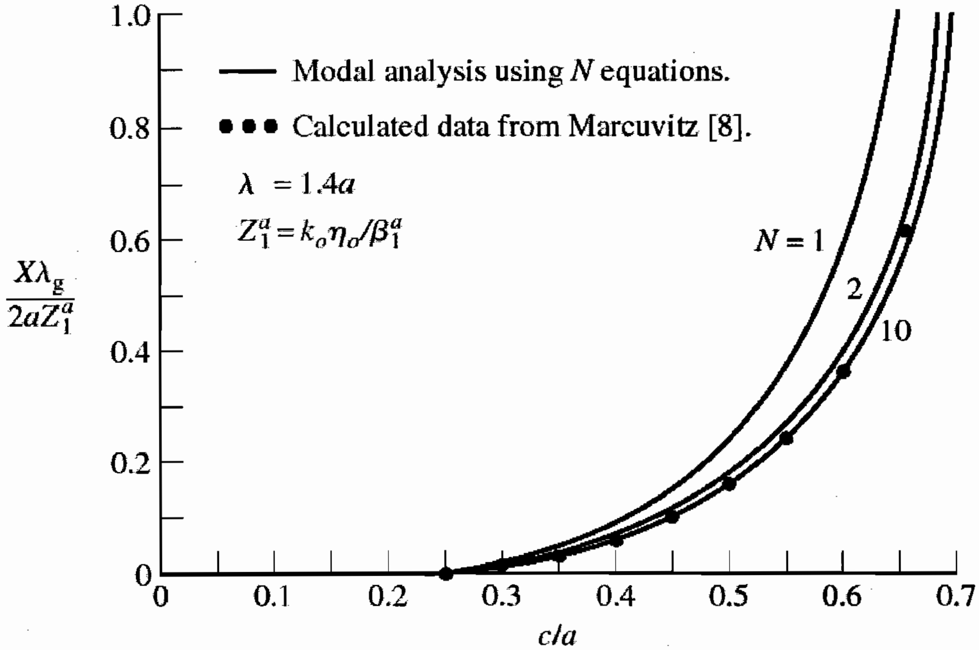


Figure 4.25 (p. 203)
Equivalent inductance of an H-plane asymmetric step.