A step discontinuity in a waveguide is an abrupt change in a waveguide dimension, but the waveguide shape doesn't change. Examples:

Can use these types of structures to make materials measurements. A good example is reduced aperture fixtures. Take rectangular waveguide:

Nice arrangement for measuring specimens that are smaller than cross section of waveguide ⇒ obtain "lower" freq. response of material than would obtain if width cross section was uniformly reduced to size of specimen.
Another arrangement provides a "open" termination (Cala Baker Jones et al.):

Rectangular waveguide

PEC

Finite mode propagation. Specimen. No prop. modes

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**H-Plane Step Discontinuity**

The first step discontinuity we'll consider is sketched below. (See Pozar, pp. 199-203).

Rectangular waveguide

Incident wave

PEC

Guide 1

Guide 2

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How do we solve such a problem? How about using a TL model?
Can't work for a number of reasons including the fact that $\Pi = (\mathcal{E}_0, z - \mathcal{E}_0, 0) / (\mathcal{E}_0, z - \mathcal{E}_0, z)$ was derived assuming $\mathcal{E}_{\text{in}} : \mathcal{F}_{\text{ii}}$ continuous at $z = 0$. No where in the b.c. $\mathcal{E}_{\text{in}} = 0$ at $z = 0$ for $c \leq x \leq a \quad 0 \leq y \leq b$, enforced that

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**TE Modal Fields at a Step Discontinuity**

The mode matching method is an excellent method for solving certain types of waveguide problems, including step discontinuities. First step is to determine the form of the fields at the step discontinuity.

We'll assumeguide 1 supports only single mode propagation and a TE$_{10}$ mode is incident on the discontinuity:

$$ E_x^1 = 0 = H_y^1 $$

(1)

$$ E_y^1 = -j \frac{\omega \mu_0}{(\beta_{c,10})^2} \cdot 1 \cdot \sin \left( \frac{\pi x}{a} \right) e^{-j \beta_{c,10} z} $$

(2)

$$ H_x^1 = \frac{j \beta_{c,10} \mu_0}{(\beta_{c,10})^2} \cdot 1 \cdot \sin \left( \frac{\pi x}{a} \right) e^{-j \beta_{c,10} z} $$

(3)

To make things a bit simpler, let's set $E_y^1$ amplitude to 1 and use TE mode impedance to express $H_x^1$ in terms of $E_y^1$:

$$ E_y^1 = \sin \left( \frac{\pi x}{a} \right) e^{-j \beta_{c,10} z} $$

(4)
We'll first look at the TE \textit{incoupled} modes (i.e., reflected + transmitted modes). Use Poise (3.829-4) and even normalisation in (4). For reflected modes replace \( r \rightarrow -r \) and

\[
E_x = \frac{\rho_x}{(\omega_0 p)^2} = -\frac{E}{X} = \frac{E}{r} \quad \text{and} \quad H_y = -\frac{E}{X} = \frac{E}{r}.
\]

For incident modes, use above.

We have all of them! (Whatever is not used will drop out.)

\[ R \mathbf{E} = \frac{R \mathbf{E}}{R \mathbf{E}} = \frac{E_X}{\rho_0} \quad \text{and} \quad \frac{R \mathbf{H}}{R \mathbf{H}} = \frac{\mathbf{H}_X}{\rho_0} \sin(k_x a) \phi \]
Applying (11) - (17) for reflected fields we find that

\[ E_y^r = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) e^{+j\beta_z z} \]  

(10)

We will introduce some notation to simplify writing these fields.

We have chosen the ref. $E_y$ of the TE mode to be an infinite summation. How much of each mode contributes to total $E_y$ we'll need to determine from $A_{mn}$.

\[ \sum_{m,n=0}^{\infty} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \]  

- $\sin^a x = \sin \left( \frac{m\pi x}{a} \right)$
- $\cos^b y = \cos \left( \frac{n\pi y}{b} \right)$

Hence,

\[ E_y^r = \sum_{m,n=0}^{\infty} A_{mn} \sin^a x \cos^b y e^{+j\beta_z z} \]  

(11)

from (8):

\[ H_x^r = \sum_{m,n=0}^{\infty} \frac{(-\beta_x^a)}{\beta_x} A_{mn} \sin^a x \cos^b y e^{+j\beta_z z} \]  

(12)

from (7):

\[ E_x^r = \sum_{m,n=0}^{\infty} \frac{-\beta_x^a}{\beta_x} A_{mn} \cos^a x \sin^b y e^{+j\beta_z z} \]  

(13)
From (9): 
\[ H_y^+ = \sum_{m=0}^{\infty} \frac{-\beta_y^m}{\beta_x^m} \exp \left( j \frac{\beta_y^m z}{\lambda_0} \right) \cos \theta \sin \phi \]  
(14)

For guide 2, the modes propagate in +z direction. We again use an infinite summation of all TE\textsuperscript{2}\textsuperscript{0} modes:

\[ E_y^+ = \sum_{p,q=0}^{\infty} B_{pq}^+ \sin \phi \cos \theta \exp \left( -j \frac{\beta_y^p z}{\lambda_0} \right) \]  
(15)

From (6): 
\[ H_x^+ = \sum_{p,q=0}^{\infty} -\frac{\beta_y^p}{\beta_x^p} B_{pq}^+ \sin \phi \cos \theta \exp \left( -j \frac{\beta_y^p z}{\lambda_0} \right) \]  
(16)

From (11): 
\[ E_x^+ = \sum_{p,q=0}^{\infty} -\frac{\beta_y^p}{\beta_x^p} B_{pq}^+ \cos \phi \sin \theta \exp \left( -j \frac{\beta_y^p z}{\lambda_0} \right) \]  
(17)

From (9) 
\[ H_y^+ = \sum_{p,q=0}^{\infty} -\frac{\beta_y^p \beta_x^q}{\beta_x^p} B_{pq}^+ \cos \phi \sin \theta \exp \left( -j \frac{\beta_y^p z}{\lambda_0} \right) \]  
(18)

The coeff of the transmitted modes are labeled B\textsubscript{pq} to distinguish them from the ref. mode coeffs. Ann. The indices p, q were chosen for the same reason.

The objective of the mode matching method is to determine the coeffs of all the "scattered" modes. Ann. & B\textsubscript{pq} for TE\textsuperscript{2}\textsuperscript{0} ref. + transmitted modes plus, in general, the coeffs of the ref. + transmitted TM\textsuperscript{2}\textsuperscript{0} modes.
Problem Simplification

These unknown coefficients are determined by enforcing the tangential boundary conditions at the discontinuity:

\begin{align}
\hat{\mathbf{z}} \times \mathbf{E}_1 &= \hat{\mathbf{z}} \times \mathbf{E}_2 \quad \text{at} \ z = 0, \ 0 \leq x \leq c, \ 0 \leq y \leq b \\
\hat{\mathbf{z}} \times \mathbf{E}_1 &= 0 \quad \text{at} \ z = 0, \ c \leq x \leq a, \ 0 \leq y \leq b \\
\hat{\mathbf{z}} \times \mathbf{H}_1 &= \hat{\mathbf{z}} \times \mathbf{H}_2 \quad \text{at} \ z = 0, \ 0 \leq x \leq c, \ 0 \leq y \leq b
\end{align}

These fields are the total fields in the waveguides on either side of the discontinuity. We cannot apply b.c.'s one mode at a time. This is how the modes "couple" together at an interface.

Before proceeding, it is very worthwhile to carefully observe the structure of the discontinuity and the incident field. This structure, and possibly symmetry, can greatly simplify mode matching problems. Sometimes

For the H-plane step, we'll show that because there is no \( y \)-variation in the incident field, the waveguides, or the discontinuity:

- Only TE\( m_0 \) modes are scattered
- No TM\( n \) modes are scattered

This will obviously greatly simplify the analysis of this problem, let's see how to discuss these outcomes.
Let's consider the b.c. on the PEC portion of the discontinuity:

\[ E_x = E_y = 0 \quad \text{for} \quad c \leq x \leq a, \quad 0 \leq y \leq b, \quad z = 0. \]

Let's look at \( E_y = E_y^b + E_y^r \) for TE \( _2 \) modes: legitimate?

\[ E_y = 0 = \sin \left( \frac{m \pi x}{a} \right) + \sum_{m,n=0}^{\infty} A_{mn} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi b}{b} \right) \]  \hspace{1cm} (22)

This is a double infinite series so perhaps not obvious how to move forward here in making simplifying statements.

We will use orthogonality properties of the sine & cosine fts.

Multiply (22) by \( \cos \left( \frac{n \pi b y}{b} \right) \) and integrate from 0 to b:

\[ 0 = \int_0^b \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi b y}{b} \right) \, dy + \int_0^b \cos \left( \frac{n \pi b y}{b} \right) \left[ \sum_{m,n=0}^{\infty} A_{mn} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi b y}{b} \right) \right] \, dy \]  \hspace{1cm} (23)

Can interchange order of the operators integral & summation since both linear and integrand not singular anywhere in domain of integration.
\[ 0 = \sin\left(\frac{\pi y}{a}\right) \int_0^b \cos\left(\frac{\pi y x}{b}\right) \, dx + \sum_{m=0}^{\infty} A_{mn} \sin\left(\frac{\pi y x}{a}\right) \int_0^b \cos\left(\frac{\pi y x}{b}\right) \, dx \]

(24)

We know that:
\[ \int_0^a \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{a}\right) \, dx = 0 \quad m \neq n \]
\[ \int_0^a \cos^2\left(\frac{\pi x}{a}\right) \, dx = \frac{a}{2} \quad m = n \]

Given this, we have:
\[ 0 = \sin\left(\frac{\pi y}{a}\right) \int_0^b \cos\left(\frac{\pi y x}{b}\right) \, dx + \sum_{m=0}^{\infty} A_{mn} \sin\left(\frac{\pi y x}{a}\right) \cdot \frac{b}{2} \]

(25)

Further,
\[ \int_0^b \cos\left(\frac{\pi y x}{b}\right) \, dx = \begin{cases} b & n' = 0 \\ 0 & n' \neq 0 \end{cases} \]

So that (25) becomes for \( n' \neq 0 \):

\[ 0 = \sum_{m=0}^{\infty} A_{mn'} \sin\left(\frac{\pi y x}{a}\right) \]

(26)

Conclude that \( A_{mn'} = 0 \) for \( n' \neq 0 \). In other words, the only coefficients that may not be zero are \( A_{m0} \Rightarrow \text{NO TE}_{mn} \) \( n \neq 0 \) modes, i.e., only \( \text{TE}^2 \) modes are \( \text{TE}_{mp} \).
Another way to view this conclusion is that with n ≠ 0, the TE\(^2\) modes all have a spatial variation in the y direction because of \(\cos(\frac{m\pi}{2L})\) in Eq. \(1\) \(H_y\) and the \(\sin(\frac{m\pi}{2L})\) variation in \(E_x\) \(H_y\).

Neither the incident field nor the step discontinuity have any variation in the y direction (within the waveguide), so the scattered fields shouldn't either. With TEM\(_m0\) modes, there's no spatial variation in y.

Lastly, we can use this same argument to show that all TM\(^2\) modes must have variation in the y direction (remember, for TM\(_m0\) modes, neither m nor n can be zero).