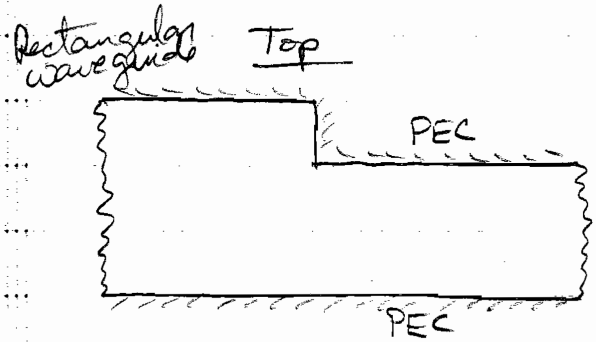
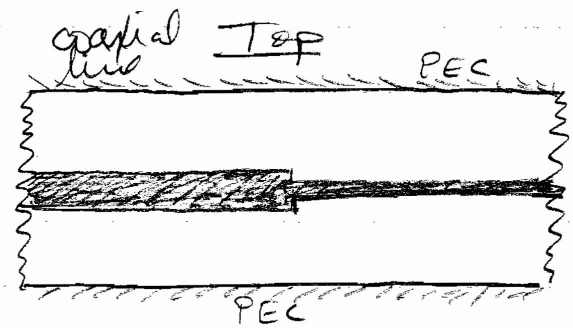
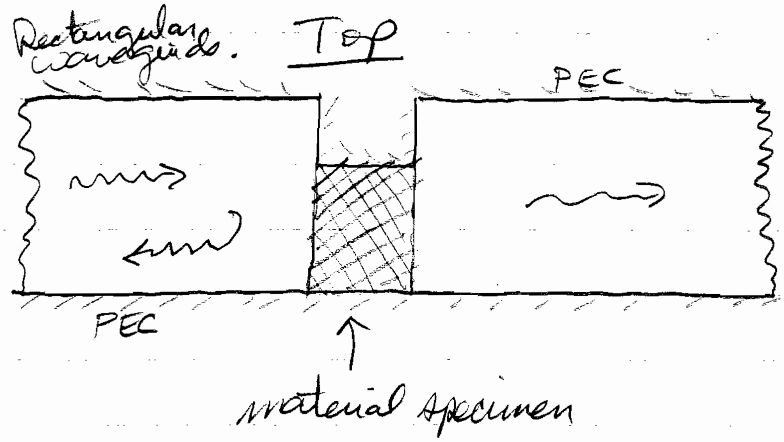
Step Discontinuities in Rectangular Waveguides

A step discontinuity in a waveguide is an abrupt change in a waveguide dimension, but the waveguide shape doesn't change. Examples:



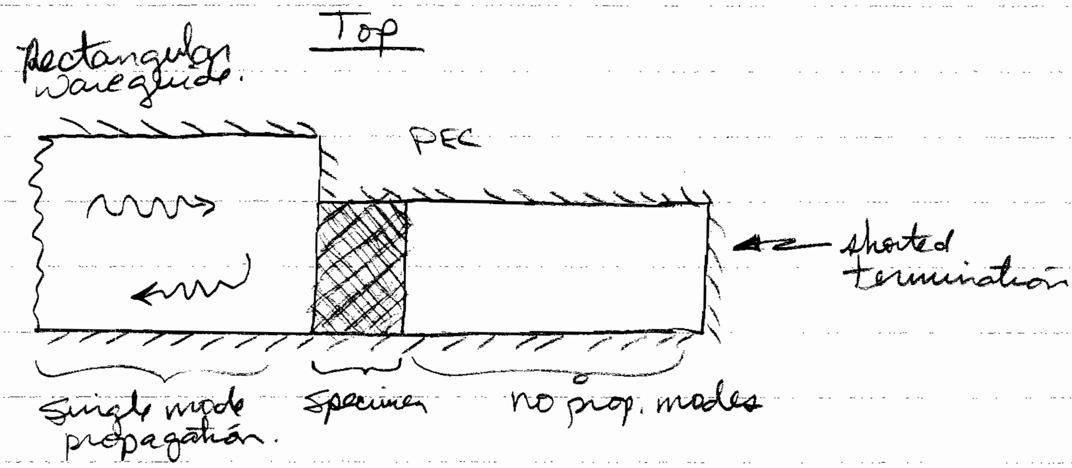


Can use these types of structures to make material measurements. A good example is reduced aperture fixtures. Take rectangular waveguide:



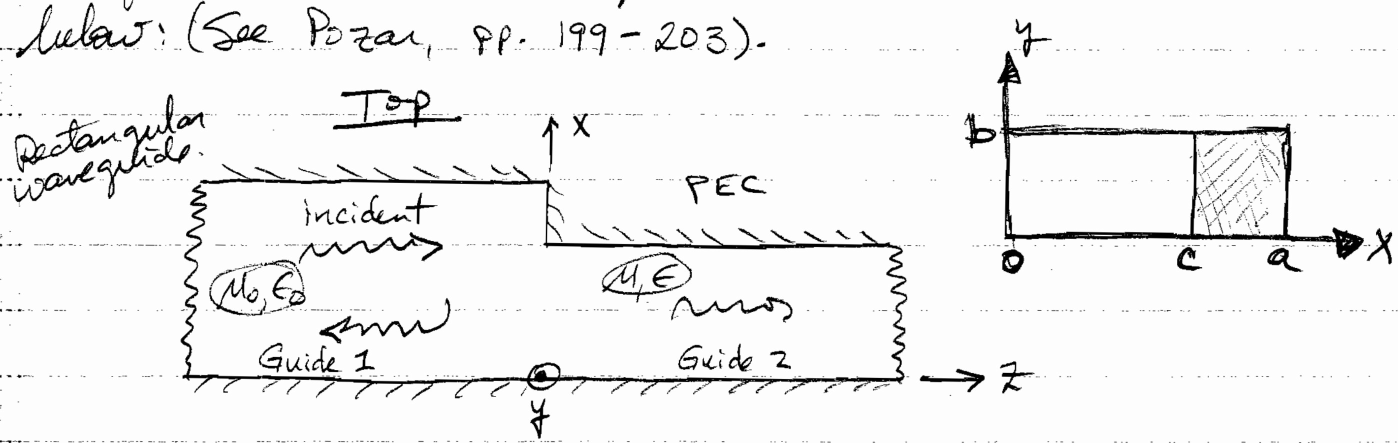
Nice arrangement for measuring specimens that are smaller than cross section of waveguide ⇒ obtain "lower" freq. response of material than would obtain if wgd cross section was uniformly reduced to size of specimen.

Another arrangement provides a "open" termination (à la Baker Jarvis et al.):

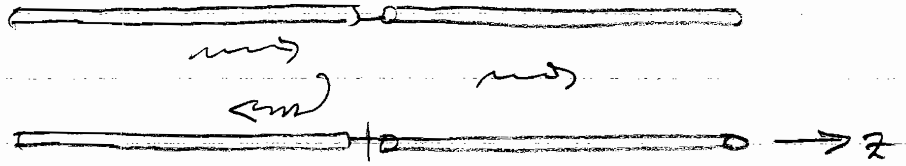


H-Plane Step Discontinuity

The first step discontinuity we'll consider is sketched below: (See Pozar, pp. 199–203).



How do we solve such a problem? How about using a TL model?



Can't work for a number of reasons including the fact that $\Gamma = (Z_{0,2} - Z_{0,1})/(Z_{0,2} + Z_{0,1})$ was derived assuming $\bar{E}_{tan}$ & $\bar{H}_{tan}$ continuous at $z=0$. No where is the b.c. enforced that $\bar{E}_{tan}=0$ at $z=0$ for $c \le x \le a$ & $0 \le y \le b$.

TE Modal Fields at a Step Discontinuity

The mode matching method is an excellent method for solving certain types of waveguide problems, including step discontinuities. First step is to determine the form of the fields at the step discontinuity.

We'll assume guide 1 supports only single mode propagation and a TE_{10} mode is incident on the discontinuity:

$$E_x^i = 0 = H_y^i \tag{1}$$

$$E_y^i = \frac{-j\omega\mu_0 \pi}{(\beta_{c,10}^a)^2 a} \cdot 1 \cdot \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_{z,10}^a z} \tag{2}$$

$$H_x^i = \frac{j\beta_{z,10}^a \pi}{(\beta_{c,10}^a)^2 a} \cdot 1 \cdot \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_{z,10}^a z} \tag{3}$$

To make things a bit simpler, let's set E_y^i amplitude to 1 & use TE mode impedance to express H_x^i in terms of E_y^i:

$$E_y^i = \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_{z,10}^a z} \tag{4}$$

and $$H_x^i = \frac{-E_x^i}{(\omega\mu_0/\beta^a_{z,10})} = \frac{-\beta^a_{z,10}}{\omega\mu_0}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta^a_{z,10}} \quad (5)$$

because
$$Z_{TE} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta_z} \quad (6)$$

for a wave prop. in the $+z$ direction. Of course, w/ the normalization in (4) & (5), H_z will no longer have an amplitude $= 1$ as we used in previous lectures.

What about reflected and transmitted fields? Which modes, types of modes, etc., do we use? Don't know, so use <u>all</u> of them! (Whatever is not needed will "drop out" from the mathematics.)

We'll first look at the TE^z "scattered" modes (i.e., the reflected & transmitted modes). Use Pozar (3.82a-d) and our normalization in (4). For reflected modes, they're prop. in $-z$ direction. Can use (3.82a-d), but replace $\beta_z^a \rightarrow -\beta_z^a$:

From $E_y^i = A_{mn}$: $\frac{-j\omega\mu}{(\beta_c^a)^2}\frac{m\pi}{a} \Rightarrow 1$

$\beta_x = \frac{m\pi}{a}$, $\beta_y = \frac{n\pi}{b}$

$\therefore$ for $+z$ prop.: $$E_x = \frac{-\beta_y}{\beta_x}A_{mn}\cos(\beta_x x)\sin(\beta_y y) \quad (7)$$

(3.86) $\rightarrow$ $$H_x = \frac{-E_y}{Z_{TE}} = \frac{-\beta_z}{\omega\mu}E_y \quad (8)$$

$$H_y = \frac{E_x}{Z_{TE}} = \frac{\beta_z}{\omega\mu}E_x \quad (9)$$

Applying (7) - (9) for reflected fields we find that

$$E_y^r = \sum_{m=0}^{\infty} \sum_{\substack{n=0 \\ (m=n\neq 0)}}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{+j\beta_z^a z} \tag{10}$$

prop in $-z$.

Will introduce some notation to simplify writing these fields.

We have chosen the ref. E_y of the TE mode to be an infinite summation. How "much" of each mode contributes to total E_y we'll need to determine from A_{mn}.

- $\sum_{m,n=0}^{\infty}{}' = \sum_{m=0}^{\infty} \sum_{\substack{n=0 \\ (m=n\neq 0)}}^{\infty}$
- $\sin_x^a = \sin\left(\frac{m\pi x}{a}\right)$
- $\cos_y^b = \cos\left(\frac{n\pi y}{b}\right)$

Hence,

$$E_y^r = \sum_{m,n=0}^{\infty}{}' A_{mn} \sin_x^a \cos_y^b\, e^{+j\beta_z^a z} \tag{11}$$

from (8): $$H_x^r = \sum_{m=n=0}^{\infty}{}' \frac{-(-\beta_z^a)}{\omega\mu_0} A_{mn} \sin_x^a \cos_y^b\, e^{+j\beta_z^a z} \tag{12}$$

from (7): $$E_x^r = \sum_{m=n=0}^{\infty}{}' -\frac{\beta_y^a}{\beta_x^a} A_{mn} \cos_x^a \sin_y^b\, e^{+j\beta_z^a z} \tag{13}$$

from (9): $$H_y^r = \sum_{m=n=0}^{\infty}{}' \frac{-\beta_y^a(-\beta_z^a)}{\beta_x^a\,\omega\mu_0} A_{mn} \cos_x^a \sin_y^b e^{+j\beta_z^a z} \quad (14)$$

For guide 2, the modes propagate in $+z$ direction. We again use an infinite summation of all TE^z modes:

$$E_y^t = \sum_{p,q=0}^{\infty}{}' B_{pq} \sin_x^c \cos_y^b e^{-j\beta_z^c z} \quad (15)$$

from (8): $$H_x^t = \sum_{p,q=0}^{\infty}{}' \frac{-\beta_z^c}{\omega\mu} B_{pq} \sin_x^c \cos_y^b e^{-j\beta_z^c z} \quad (16)$$

from (7): $$E_x^t = \sum_{p,q=0}^{\infty}{}' -\frac{\beta_y^c}{\beta_x^c} B_{pq} \cos_x^c \sin_y^b e^{-j\beta_z^c z} \quad (17)$$

from (9) $$H_y^t = \sum_{p,q=0}^{\infty}{}' \frac{-\beta_y^c\beta_z^c}{\omega\mu\,\beta_x^c} B_{pq} \cos_x^c \sin_y^b e^{-j\beta_z^c z} \quad (18)$$

The coeff. of the transmitted modes are labeled B_{pq} to distinguish them from the ref. mode coeffs. A_{mn}. The indices p, q were chosen for the same reason.

The objective of the mode matching method is to determine the coeffs of all the "scattered" modes: A_{mn} & B_{pq} for TE^z ref. & transmitted modes plus, in general, the coeffs of the ref. & transmitted TM^z modes.

Problem Simplification

These unknown coeffs are determined by enforcing the tangential-field boundary conditions at the discontinuity:

$$\hat{z}\times\bar{E}_1 = \hat{z}\times\bar{E}_2 \quad \text{at } z=0,\ 0\le x\le c,\ 0\le y\le b \tag{19}$$

$$\hat{z}\times\bar{E}_1 = 0 \quad \text{at } z=0,\ c\le x\le a,\ 0\le y\le b \tag{20}$$

$$\hat{z}\times\bar{H}_1 = \hat{z}\times\bar{H}_2 \quad \text{at } z=0,\ 0\le x\le c,\ 0\le y\le b \tag{21}$$

These fields are the <u>total</u> fields in the waveguides on either side of the discontinuity. We cannot apply b.c.'s one mode @ a time. This is how the modes "couple" together at an interface.

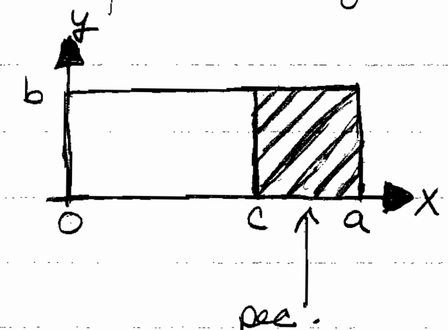
Before proceeding, it is very worthwhile to carefully observe the structure of the discontinuity and the incident field. This structure, and possibly symmetry, can sometimes greatly simplify mode matching problems.

For the H-plane step, we'll show that because there is no *y-variation in the incident field, the waveguides, or the discontinuity:

- Only TE_{m0} modes are scattered
- No TM^z modes are scattered.

This will obviously greatly simplify the analysis of this prob. Let's see how to discern these outcomes.

Let's consider the b.c. on the PEC portion of the discontinuity.



$$E_x = E_y = 0 \quad \text{for } c \le x \le a,\; 0 \le y \le b,\; z = 0.$$

Let's look at $E_y = E_y^i + E_y^r$ for TE^z modes : legitimate?

$$E_y = 0 = \sin\left(\frac{\pi x}{a}\right) + \sum_{m,n=0}^{\infty}{}' A_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad (22)$$

This is a doubly infinite series, so perhaps not obvious how to move forward here in making simplifying statements.

We will use orthogonality properties of the sine & cosine fcts.

Multiply (22) by $\cos\left(\frac{n'\pi y}{b}\right)$ and integrate from 0 to b :

$$0 = \int_0^b \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{n'\pi y}{b}\right) dy + \int_0^b \cos\left(\frac{n'\pi y}{b}\right)\left[\sum_{m,n=0}^{\infty}{}' A_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\right] dy \qquad (23)$$

Can interchange order of the operators integral & summation since both linear and integrand not singular anywhere in domain of integration.

$$0 = \sin\left(\frac{\pi x}{a}\right)\int_0^b \cos\left(\frac{n'\pi y}{b}\right)dy + \sum_{m,n=0}^{\infty}{}' A_{mn}\sin\left(\frac{m\pi x}{a}\right)\int_0^b \cos\left(\frac{n'\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)dy \quad (24)$$

We know that: $\int_0^a \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right)dx = 0 \quad m \neq n$

$\int_0^a \cos^2\left(\frac{n\pi x}{a}\right)dx = \frac{a}{2}$ $\quad m = n$

Giving

$$0 = \sin\left(\frac{\pi x}{a}\right)\int_0^b \cos\left(\frac{n'\pi y}{b}\right)dy + \sum_{m=0}^{\infty}{}' A_{mn'}\sin\left(\frac{m\pi x}{a}\right)\cdot\frac{b}{2} \quad (25)$$

Further, $\int_0^b \cos\left(\frac{n'\pi y}{b}\right)dy = \begin{cases} b & n' = 0 \\ 0 & n' \neq 0 \end{cases}$

So that (25) becomes for $\underline{n' \neq 0}$:

$$0 = \sum_{m=0}^{\infty}{}' A_{mn'}\sin\left(\frac{m\pi x}{a}\right) \quad (26)$$

Conclude that $A_{mn'} = 0$ for $n' \neq 0$. In other words, the only coeffs that may not be zero are $A_{m0} \Rightarrow$ No TE_{mn} $n \neq 0$ modes, i.e., only TE^z modes are TE_{m0}.

Another way to view this conclusion is that with $n \neq 0$, the TE^z modes all have a spatial variation in the y direction because of $\cos(\frac{n\pi y}{b})$ in E_y & H_x and the $\sin(\frac{n\pi y}{b})$ variation in E_x & H_y.

Neither the incident field nor the step discontinuity have any variation in the y direction (within the waveguide), so the scattered fields shouldn't either. With TE_{m0} modes, there's no spatial variation in y.

Lastly, we can use this same argument to show that all TM^z modes must have variation in the y direction (remember, for TM_{mn} modes, neither m nor n can be zero).