Orthogonality Properties of Modes in Waveguides

We have determined solutions for TE^z & TM^z modal solutions in rectangular waveguides. For finite frequency, only a finite number of these modes can propagate down the wvgd and carry power.

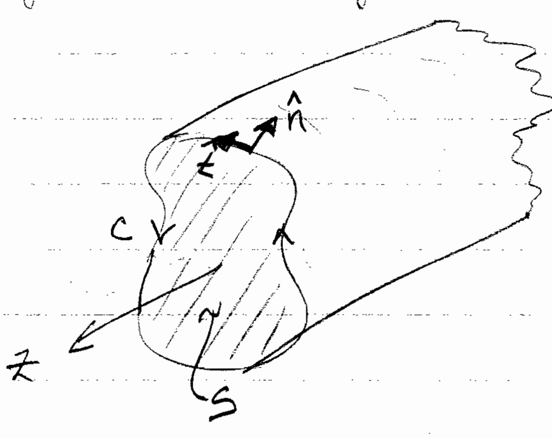
We'll show in this lecture that the power carried by the EM wave is divided onto these prop. modes, of which only the $\bar{E}$ & $\bar{H}$ of a specific mode interact to carry this power. In other words, modes in homogeneous wvgds are orthogonal in the sense that the $\bar{E}$ & $\bar{H}$ of different modes do not transport energy

---

Ref.: Collin, Field Theory of Guided Waves.

Orthogonality of Axial fields

Consider a homogeneous waveguide with PEC walls.



$\hat{n} \times \hat{t} = \hat{z}$

We found previously that the axial field comps E_z or H_z satisfy the reduced wave eqn

$$\nabla_t^2 \Psi_i + \beta_{ci}^2 \Psi_i = 0 \qquad (1)$$

where $\Psi_i(x,y) = e_{zi}(x,y)$ or $h_{zi}(x,y)$ for the i^{th} mode (m & n pair). Multiply (1) by Ψ_j (another mode)

$$\Psi_j \nabla_t^2 \Psi_i + \beta_{ci}^2 \Psi_i \Psi_j = 0 \qquad (2)$$

Similarly, we begin w/ the reduced wave eqn for Ψ_j and multiply by Ψ_i giving

$$\Psi_i \nabla_t^2 \Psi_j + \beta_{cj}^2 \Psi_i \Psi_j = 0 \qquad (3)$$

Subtracting (3) from (2) gives

$$\Psi_j \nabla_t^2 \Psi_i - \Psi_i \nabla_t^2 \Psi_j + (\beta_{ci}^2 - \beta_{cj}^2) \Psi_i \Psi_j = 0 \qquad (4)$$

We now integrate (4) over a plane transverse to the direction of prop. at some arbitrarily chosen z:

$$(\beta_{ci}^2 - \beta_{cj}^2) \int_S \Psi_i \Psi_j \, ds = \int_S (\Psi_i \nabla_t^2 \Psi_j - \Psi_j \nabla_t^2 \Psi_i) \, ds \qquad (5)$$

We will next employ Green's Second Identity to the RHS of (5). This is an identity that provides a mathematical statement of reciprocity. In three dimensions:

$$\oint_{S(V)} \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) ds = \int_{V(S)} (\psi \nabla^2 \phi - \phi \nabla^2 \psi) \, dv \qquad (6)$$

where ϕ & ψ are scalar fields with suitable differentiation properties, & closed surface S encloses volume V, & $\hat{n}$ is the outward unit normal.

In 2-D, Green's second identity is

$$\oint_{C(S)} \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) dl = \int_{S(C)} \left(\psi \nabla_t^2 \phi - \phi \nabla_t^2 \psi\right) ds \qquad (7)$$

Applying (7) to RHS of (5) gives

$$(\beta_{c_i}^2 - \beta_{c_j}^2) \int_S \psi_i \psi_j \, ds = \oint_{C(S)} \left(\psi_i \frac{\partial \psi_j}{\partial n} - \psi_j \frac{\partial \psi_i}{\partial n}\right) dl \qquad (8)$$

This eqn. is important. Notice that for TM^z modes, ψ_i & ψ_j ($= e_z$) vanish along C because of PEC walls. Hence, for TE^z modes, (8) becomes

$$(\beta_{c_i}^2 - \beta_{c_j}^2) \int_S \psi_i \psi_j \, ds = 0 \qquad (9)$$

for $i \neq j \Rightarrow$ $\underline{\int_S \psi_i \psi_j \, ds = 0}$ provided $\beta_{c_i}^2 \neq \beta_{c_j}^2$ (10)

which means that the axial components of TE^z modes are orthogonal (in a functional sense).

What about TE^z modes? It can be shown that for these modes

$$\frac{\partial \psi}{\partial n} = 0 \qquad (11)$$

Using (11) in (9) we arrive at the same eqn. (10) so that the axial comps. of TE^z modes are also orthogonal.

can show that in the case of degenerate modes in (9) where $\beta_{ci}^2 = \beta_{cj}^2$ (but not same mode, of course) the orthogonality relationship (10) is still valid.

Orthogonality of Transverse Fields

It is also possible to show that the transverse $\bar{E}$ of one mode & transverse $\bar{H}$ of another are orthogonal (in the functional sense) over the cross-section of a wvgd.

Let $\bar{E}_{ti}$, $\bar{H}_{ti}$ and $\bar{E}_{tj}$, $\bar{H}_{tj}$ be the transverse $\bar{E}$ & $\bar{H}$ of modes i & j. From Faraday's law:

$$\nabla \times \bar{E}_i = -j\omega\mu \bar{H}_i \quad \& \quad \nabla \times \bar{E}_j = -j\omega\mu \bar{H}_j \qquad (12)$$

Dot multiply first by $\bar{H}_j$ & second by $\bar{H}_i$ and subtract

$$\bar{H}_i \cdot \nabla \times \bar{E}_j - \bar{H}_j \cdot \nabla \times \bar{E}_i = 0 \qquad (13)$$

Similarly, from Ampere's law

$$\bar{E}_i \cdot \nabla \times \bar{H}_j - \bar{E}_j \cdot \nabla \times \bar{H}_i = 0 \qquad (14)$$

Adding (13) & (14):

$$\bar{H}_i \cdot \nabla \times \bar{E}_j - \bar{E}_j \nabla \times \bar{H}_i + \bar{E}_i \cdot \nabla \times \bar{H}_j - \bar{H}_j \cdot \nabla \times \bar{E}_i = 0 \qquad (15)$$

Applying the vector i.d. $\nabla\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot\nabla\times\bar{A} - \bar{A}\cdot(\nabla\times\bar{B})$ (16) to the first two & last two terms in (15) gives

$$\nabla\cdot(\bar{E}_j\times\bar{H}_i) - \nabla\cdot(\bar{H}_j\times\bar{E}_i) = 0$$

or

$$\nabla\cdot(\bar{E}_j\times\bar{H}_i - \bar{H}_j\times\bar{E}_i) = 0 \qquad (17)$$

For wave prop in $+z$ s.t. $\bar{E}_i, \bar{H}_i \propto e^{-\gamma_i z}$ & $\bar{E}_j$ & $\bar{H}_j \propto e^{-\gamma_j z}$ then (17) becomes

$$\nabla\cdot(\bar{E}_j\times\bar{H}_i - \bar{H}_j\times\bar{E}_i) = \nabla_t\cdot(\bar{E}_j\times\bar{H}_i - \bar{H}_j\times\bar{E}_i) + \hat{z}\cdot\frac{\partial}{\partial z}(\bar{E}_j\times\bar{H}_i - \bar{H}_j\times\bar{E}_i)$$

$$= \nabla_t\cdot(\bar{E}_j\times\bar{H}_i - \bar{H}_j\times\bar{E}_i) - (\gamma_i+\gamma_j)\,\hat{z}\cdot(\bar{E}_{tj}\times\bar{H}_{ti} - \bar{E}_{ti}\times\bar{H}_{tj}) = 0 \qquad (18)$$

Only transverse fields because $\hat{z}\cdot$ others $= 0$.

alntegrating (18) over a cross-sectional plane gives

$$\int_S \nabla_t\cdot(\bar{E}_j\times\bar{H}_i - \bar{H}_j\times\bar{E}_i)\,ds = (\gamma_i+\gamma_j)\int_S \hat{z}\cdot(\bar{E}_{tj}\times\bar{H}_{ti} - \bar{E}_{ti}\times\bar{H}_{tj}) \qquad (19)$$

We can now apply the 2-D form of the divergence theorem to the LHS of (19)

$$\int_{S(C)} \nabla_t \cdot (\bar{E}_j \times \bar{H}_i - \bar{H}_j \times \bar{E}_i)\, ds = \oint_{C(S)} \hat{n} \cdot (\bar{E}_j \times \bar{H}_i - \bar{H}_j \times \bar{E}_i)\, dl. \quad (20a)$$

which using the vector i.d. $\bar{A} \cdot (\bar{B} \times \bar{C}) = \bar{C} \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\bar{C} \times \bar{A})$ becomes

$$= \oint_{C(S)} \left[\bar{H}_i \cdot (\hat{n} \times \bar{E}_j) - \bar{B} \cdot (\bar{E}_i \times \hat{n}) \right] dl \quad (20b)$$

On the PEC walls of the guide, $\hat{n} \times \bar{E} = 0$ so the RHS in (20b) vanishes and (19) becomes

$$(\gamma_i + \gamma_j) \int_S \hat{z} \cdot (\bar{E}_{tj} \times \bar{H}_{ti} - \bar{E}_{ti} \times \bar{H}_{tj})\, ds = 0 \quad (21)$$

or equivalently

$$(\gamma_i + \gamma_j) \int_S \hat{z} \cdot (\bar{e}_j \times \bar{h}_i - \bar{e}_i \times \bar{h}_j)\, ds = 0 \quad (22)$$

This is very close to our final eqn. for transverse field orthogonality, though what we're interested in is not the difference in the integrand in (22) but in each term separately.

We can show that each term in (22) vanishes separately. To accomplish this, we will also consider mode $\bar{E}_i'$, $\bar{H}_i'$ which is same mode as i but prop. in $-\hat{z}$ direction as $e^{\gamma_i z}$.

For this i' mode, $\bar{E}'_{ti} = \bar{e}_i e^{\gamma_i z}$ & $\bar{H}'_{ti} = -h_i e^{\gamma_i z}$ (transverse E same but transverse H changes sign — think power flow).

So, if we begin with (12) for $\bar{E}_i', \bar{H}_i'$ and $\bar{E}_j$ & $\bar{H}_j$ and repeat the steps leading to (22) we find that

$$(\gamma_j - \gamma_i)\int_S \hat{z}\cdot\left(-\bar{e}_j \times \bar{h}_i - \bar{e}_i \times \bar{h}_j\right) ds = 0 \qquad (23)$$

Adding (22) & (23) we find that

$$\int_S (\bar{e}_i \times \bar{h}_j)\cdot\hat{z}\, ds = 0 \qquad (24)$$

while subtracting

$$\int_S (\bar{e}_j \times \bar{h}_i)\cdot\hat{z}\, ds = 0 \qquad (25)$$

This is the desired orthogonality relation for transverse fields in a homogeneous wvgd.

Power Flow

We can repeat this derivation beginning w/ (12) for fields $\bar{E}_i, \bar{H}_i$ and $\bar{E}_j^*, \bar{H}_j^*$ where '*' is complex conjugate.

The resulting orthogonality statement analogous to (24) can be found as

$$\int_S (\bar{e}_i \times \bar{h}_j^*) \cdot \hat{z} \, ds = 0 \qquad (26)$$

for PEC walls and lossless material in the waveguide (such that $\epsilon = \epsilon^*$ & $\mu = \mu^*$).