Similar to TEM waves, it is possible to identify transmission line equivalent representations for TE and TM modes in waveguides. While this is possible, there are important differences in the physical interpretation of equivalent voltage, current, and impedance.

In particular, for waveguides:
- Voltage and current are defined for a specific waveguide mode \((m,n)\), where the equivalent voltage \(V\) is proportional to the transverse electric field, and the equivalent current \(I\) is proportional to the transverse magnetic field.
- The ratio of \(V\) to \(I\) for a \(z \pm 2\) propagating mode equals the wave impedance of the waveguide.
- Are there such a scalar quantity? Need to see.
- The product \(zV \cdot I^*\) should yield power propagating in \(z\) direction, respectively, in waveguide for a specific propagating mode.

Equivalent admittance

As defined above, the ratio of the transverse \(E_i / H\) should equal the wave admittance. We'll examine this for TE and TM modes in hollow metal waveguide.
For TE modes: We've shown previously for $e^{-j\beta z}$ propagation that

\[
E_x = -\frac{j \omega \mu}{\beta_c^2} \frac{\partial H_z}{\partial y} \quad (1) \quad E_y = \frac{j \omega \mu}{\beta_c^2} \frac{\partial H_z}{\partial x} \quad (3)
\]

\[
H_y = -\frac{j \beta_c^2}{\beta_c^2} \frac{\partial H_z}{\partial y} \quad (2) \quad H_x = -\frac{j \beta_c^2}{\beta_c^2} \frac{\partial H_z}{\partial x} \quad (4)
\]

Dividing (1) by (2) for $\frac{\partial H_z}{\partial y} \neq 0$:

\[
\frac{E_x}{H_y} = \frac{\omega \mu}{\beta_c^2} \cdot \frac{\beta_c^2}{\beta_c^2} = \frac{\omega \mu}{\beta_c^2} \equiv \frac{1}{Z_{TE}}
\]

and dividing (3) by (4) for $\frac{\partial H_z}{\partial x} \neq 0$:

\[
\frac{E_y}{H_x} = \frac{\omega \mu}{\beta_c^2} \cdot \frac{(\beta_c^2)}{\beta_c^2} = -\frac{\omega \mu}{\beta_c^2} \equiv -\frac{1}{Z_{TE}}
\]

More compactly, we can express (5) and (6) in this one equation

\[
\hat{h}_{TE} = \frac{1}{Z_{TE}} \hat{\mathbf{x}} \times \hat{\mathbf{e}}_{TE}
\]

\[
\left[ = \frac{1}{Z_{TE}} \hat{\mathbf{x}} \left( \hat{\mathbf{x}} e_{x,TE} + \hat{\mathbf{y}} e_{y,TE} \right) \right] \times \left[ = \frac{1}{Z_{TE}} \hat{\mathbf{y}} e_{x,TE} - \hat{\mathbf{x}} \right]
\]

This mode impedance $Z_{TE} = \frac{\omega \mu}{\beta_c^2}$ is a function of frequency, shape, as well as the mode index $m,n$.

For TM modes: We've shown previously for $e^{-j\beta z}$ propagation that
\[ E_x = -i \frac{\beta_x}{\beta_e} \frac{\partial E_y}{\partial x} \quad (8) \]
\[ E_y = -i \frac{\beta_x}{\beta_e^2} \frac{\partial E_x}{\partial y} \quad (10) \]
\[ H_y = -i \frac{\omega e}{\beta_e} \frac{\partial E_x}{\partial x} \quad (9) \]
\[ H_x = i \frac{\omega e}{\beta_e^2} \frac{\partial E_y}{\partial y} \quad (11) \]

Dividing (8) by (9) for \( \frac{\partial E_x}{\partial x} \neq 0 \):
\[ \frac{E_x}{H_y} = -i \frac{\beta_x^2}{\beta_e^3} \frac{\beta_e}{\omega e} = -i \frac{\beta_x}{\omega e} \equiv -\pi_{TM} \quad (12) \]

While dividing (10) by (11) for \( \frac{\partial E_y}{\partial y} \neq 0 \):
\[ \frac{E_y}{H_x} = -i \frac{\beta_x^2}{\beta_e} \frac{\beta_e}{\omega e} = -i \frac{\beta_x}{\omega e} \equiv -\pi_{TM} \quad (13) \]

Eqs (12) and (13) can be expressed more compactly as
\[ \pi_{TM} = \frac{1}{\pi_{TM}} \hat{\beta} \times \vec{e}_{TM} \quad (14) \]

As for TE, this TM mode admittance is also a function of frequency and the modes include m \& n. Note that (9) and (14) have identical forms; only the modal admittance differ.
Equivalent Voltage & Currents

In the case of both +z and -z propagating waves in a waveguide, we can express the transverse $E + H$ for a specific mode $(m, n)$ as:

$$E_{\pm}(x, y, z) = \bar{E}(x, y) \left( A^+ e^{-j\beta_z z} + A^- e^{+j\beta_z z} \right)$$  \(4.6a, 15\)

and

$$H_{\pm}(x, y, z) = \bar{H}(x, y) \left( A^+ e^{-j\beta_z z} - A^- e^{+j\beta_z z} \right)$$  \(4.6b, 16\)

where we've just found that for a particular mode $(m, n)$

$$\bar{H}(x, y) = \frac{1}{\varepsilon_w} \hat{z} \times \bar{E}(x, y)$$  \(4.7, 17\)

and $\varepsilon_w = \varepsilon_{TE}$ or $\varepsilon_{TM}$ as appropriate.

From (15) we can quickly identify scalar equations that are directly proportional to transverse $E + H$ and depend only on $z$. In particular, we'll derive from (15)

$$\bar{E}_{\pm}(x, y, z) = \bar{E}(x, y) \cdot \frac{V(z)}{c_i}$$  \(18\)

while from (16)

$$\bar{H}_{\pm}(x, y, z) = \bar{H}(x, y) \cdot \frac{I(z)}{c_i}$$  \(19\)

\[V(z) = V^+ e^{-j\beta_z z} + V^- e^{+j\beta_z z}\]  \(4.8a, 20\)

and

$I(z) = I^+ e^{-j\beta_z z} - I^- e^{+j\beta_z z}$  \(4.8b, 21\)

Comparing (20), (18) & (15) we find that $\frac{V^+}{c_i} = H^+ \frac{V^-}{c_i} = H^-$
or \[ C_1 = \frac{V^+}{A^+} = \frac{V^-}{A^-}. \] Similarly, comparing (21), (19), and (16)

\[ C_2 = \frac{I^+}{A^+} = \frac{I^-}{A^-}. \] (23)

Further, the ratio \[ \frac{V^+}{I^+} \text{ is associated with the wave impedance.} \]

From (22) and (23),

\[ \frac{C_1}{C_2} = \frac{\frac{V^+}{A^+}}{\frac{I^+}{A^+}} = \frac{V^+}{I^+} = \frac{Z_W}{Q} \] (24)

Similarly from (22) and (23):

\[ \frac{C_1}{C_2} = \frac{V^-}{I^-} = Z_W \] (25)

Hence, the equivalent TL model is given for the hollow metallic waveguide as from (20) and (20) using (24) and (25).

\[ V(z) = V^+ e^{-j\beta_2 z} + V^- e^{+j\beta_2 z} \] (26)

\[ I(z) = \frac{V^+}{Z_W} e^{-j\beta_2 z} - \frac{V^-}{Z_W} e^{+j\beta_2 z} \] (27)

These TL equations are very similar to those that model TEM modes. Primary differences are:

- The TL is intrinsically dispersive because \( \beta_2 \) is a function of \( \nu_2 \).
- The TL impedance is intrinsically frequency dependent.
- This TL models a single mode \((m,n)\), propagating.
- The equivalent \( V(z) \) and \( I(z) \) are proportional to the...
Amplitude of the transverse $E$ : $H$, respectively.

This is in contrast to the TL model of TEM waves where $V$ is proportional to a line integral of $E$ in a plane transverse to $z$ and $I$ is related to a closed line integral of $H$ in the same plane enclosing one of the conductors.

Application of TL Model

The utility of the TL model for hollow metallic waveguides is that we can use it to simplify the analysis of certain waveguide problems. An example of such a problem is the joint of two homogeneously filled waveguides, as in the next example.

Example: A WR-90 waveguide is half filled of ferrodielectric as shown below. Determine the electric field reflection coefficient at $Z = 0$ using an equivalent TL model for a freq. of 8 GHz. Plot over X band.

\[ Z_{01} \rightarrow Z_{02} \]
For WR-90 waveguide, \( a = 2.286 \text{ cm (0.9")} \) and \( b = 1.016 \text{ cm (0.4")} \)

For the TE\(_{0}\) modes,

\[
f_{c,0} = \frac{1}{2\pi \sqrt{\varepsilon_r}} a = \frac{MC_0}{2a \sqrt{\varepsilon_r}}
\]

The two lowest-order modes in the two waveguides are:

- **Air (\( \varepsilon_r = 1 \))**
  - \( m=1 \): \( f_{c,10} = 6.557 \text{ GHz} \)
  - \( m=2 \): \( f_{c,20} = 13.11 \text{ GHz} \)

- **Resin (\( \varepsilon_r = 2.54 \))**
  - \( m=1 \): \( f_{c,10} = 4.11 \text{ GHz} \)
  - \( m=2 \): \( f_{c,20} = 8.23 \text{ GHz} \)

At 8 GHz, only TE\(_{10}\) mode propagates in both waveguides.

For TE\(_2\) modes, the wave number is given by \((5)\) to be

\[
x_{TE} = \frac{\omega n}{\beta z}
\]

For the equivalent TLS,

\[
z_{01} = \frac{\omega M_0}{\beta^2} \frac{\omega M_0}{\sqrt{\beta^2 n^2 - (\pi \alpha)^2}} = 657.6 \Omega
\]

and

\[
z_{02} = \frac{\omega M_0}{\sqrt{\beta^2 \varepsilon_r - (\pi \alpha)^2}} = 275.6 \Omega
\]

... Smaller than because "fatter" from cutoff.

... The electric field ref. coeff. can be computed from the equivalent TLS model in the traditional fashion as

\[
\Gamma = \frac{z_{02} - z_{01}}{z_{02} + z_{01}} = -0.41 @ 8 \text{ GHz}
\]
... (Why is the equivalent to the electric field ref. coeff. in the w/g?)

Plot of $\Pi$ vs. $f$ is shown on next page over the frequency range where single mode prop. in both guides. Very large ref @ $\nu \leq 0.6$ GHz because quick cut-off.

As $f$ increases, thinking of beam wave range, $\Pi$ should approach that of a UPW normally incident on a half space of beam. Let:

$$\Gamma_{TEM} = \frac{\eta_r - 1}{\eta_r + 1} = \frac{\sqrt{\eta_r} - 1}{\sqrt{\eta_r} + 1} = -0.229.$$
In[75] := ClearAll[ε0, μ0, GHz, a, b, βz, m, n, f, εr, ZOTE, T, εr1, εr2, freq]

In[76] := a = 0.9*2.54/100.;
    b = 0.4*2.54/100.;
    ε0 = 8.854*10^-12;
    μ0 = 4.*Pi*10^-7;
    GHz = 10^9;

In[81] := βz[m_, n_, f_, εr_] := Sqrt[(2.*Pi*f)^2*μ0*ε0*εr - (m*Pi/a)^2 - (n*Pi/b)^2]
    ZOTE[m_, n_, f_, εr_] := 2.*Pi*f*μ0/βz[m, n, f, εr]
    T[m_, n_, f_, εr1_, εr2_] :=
        (ZOTE[m, n, f, εr2] - ZOTE[m, n, f, εr1]) / (ZOTE[m, n, f, εr2] + ZOTE[m, n, f, εr1])

In[84] := εr1 = 1.;
    εr2 = 2.54;
    freq = 8.*GHz;
    βz[1, 0, freq, εr1]
    βz[1, 0, freq, εr2]
    ZOTE[1, 0, freq, εr1]
    ZOTE[1, 0, freq, εr2]
    T[1, 0, freq, εr1, εr2]

Out[87] = 96.0495
Out[88] = 229.167
Out[89] = 657.634
Out[90] = 275.63
Out[91] = -0.40932

In[100] :=
    Plot[T[1, 0, f*GHz, εr1, εr2], {f, 6.558, 8.229},
        AxesLabel -> {"f [GHz]", "\(\Gamma\)"}, PlotStyle -> {RGBColor[1, 0, 0]}, PlotRange -> All]

Out[100] = - Graphics -