Equivalent voltage, current, and impedance for waveguides

Similar to TEM waves, it is possible to identify transmission line equivalent representations for TE & TM modes in waveguides. While this is possible, there are important differences in the physical interpretation of equivalent voltage, current, and impedance.

In particular, for waveguides:

- Voltage & current are defined for a specific waveguide mode (m,n), where the equivalent voltage V for a $\pm z$ prop mode is proportional to the transverse electric field & the equivalent current I is proportional to the transverse magnetic field.
- The ratio of V to I for a $\pm z$ propagating mode equals the wave impedance in the waveguide. Is there such a scalar quantity? Need to see.
- The product $\pm\frac{1}{2}V\cdot I^*$ should yield power propagating in $\pm z$ directions, respectively, in waveguide for a specific propagating mode.

Equivalent impedance

As defined above, the ratio of the transverse E & H should equal the wave impedance. We'll examine this for TE^z & TM^z modes in hollow metallic waveguides.

- **For TE^z modes:** We've shown previously for $e^{-j\beta_z z}$ propagation that

$$E_x = \frac{-j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial y} \quad (1) \quad , \quad E_y = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x} \quad (3)$$

$$H_y = \frac{-j\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial y} \quad (2) \quad , \quad H_x = \frac{-j\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial x} \quad (4)$$

Dividing (1) by (2) for $\frac{\partial H_z}{\partial y} \neq 0$:

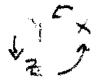
$$\frac{E_x}{H_y} = \frac{\omega\mu}{\beta_c^2}\cdot\frac{\beta_c^2}{\beta_z} = \frac{\omega\mu}{\beta_z} \equiv Z_{TE} \quad (5)$$

and dividing (3) by (4) for $\frac{\partial H_z}{\partial x} \neq 0$:

$$\frac{E_y}{H_x} = \frac{\omega\mu}{\beta_c^2}\cdot\left(-\frac{\beta_c^2}{\beta_z}\right) = \frac{-\omega\mu}{\beta_z} \equiv -Z_{TE} \quad (6)$$

More compactly, we can express (5) and (6) in this one equation

$$\bar{h}_{TE} = \frac{1}{Z_{TE}}\hat{z}\times\bar{e}_{TE} \quad (7)$$



$$\left[= \frac{1}{Z_{TE}}\hat{z}\times\left(\hat{x}e_{x,TE} + \hat{y}e_{y,TE}\right) = \frac{1}{Z_{TE}}\hat{y}e_{x,TE} - \hat{x}\frac{1}{Z_{TM}}e_{y,TE}\right] \checkmark$$

This mode impedance $Z_{TE} = \frac{\omega\mu}{\beta_z}$ is a fct of frequency, **shape,** as well as the mode indices m & n.

- **For TM^z modes:** We've shown previously for $e^{-j\beta_z z}$ propagation that

$$E_x = \frac{-j\beta_z}{\beta_c^2}\frac{\partial E_z}{\partial x} \quad (8), \qquad E_y = \frac{-j\beta_z}{\beta_c^2}\frac{\partial E_z}{\partial y} \quad (10)$$

$$H_y = \frac{-j\omega\epsilon}{\beta_c^2}\frac{\partial E_z}{\partial x} \quad (9), \qquad H_x = \frac{j\omega\epsilon}{\beta_c^2}\frac{\partial E_z}{\partial y} \quad (11)$$

Dividing (8) by (9) for $\frac{\partial E_z}{\partial x} \neq 0$:

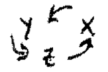
$$\frac{E_x}{H_y} = \frac{-j\beta_z}{\beta_c^2}\cdot\frac{\beta_c^2}{-j\omega\epsilon} = \frac{\beta_z}{\omega\epsilon} \equiv Z_{TM} \qquad (12)$$

while dividing (10) by (11) for $\frac{\partial E_z}{\partial y} \neq 0$:

$$\frac{E_y}{H_x} = \frac{-j\beta_z}{\beta_c^2}\cdot\frac{\beta_c^2}{j\omega\epsilon} = \frac{-\beta_z}{\omega\epsilon} \equiv -Z_{TM} \qquad (13)$$

Eqns (12) and (13) can be expressed more compactly as

$$\bar{h}_{TM} = \frac{1}{Z_{Tm}}\,\hat{z} \times \bar{e}_{Tm} \qquad (14)$$



$$\left[= \frac{1}{Z_{Tm}}\,\hat{z} \times \left(\hat{x}\,e_{x,Tm} + \hat{y}\,e_{y,Tm}\right) = \frac{1}{Z_{Tm}}\,\hat{y}\,e_{x,Tm} - \hat{x}\,e_{y,Tm} \right]$$ ✗

As for TE, this TM mode impedance is also a fct. of frequency, shape, and the mode indices m & n. Note that (7) and (14) have identical forms; only the modal impedances differ.

Equivalent Voltages & Currents

In the case of both +z and −z prop. waves in a waveguide, we can express the transverse $\bar{E}$ & $\bar{H}$ for a specific mode (m,n) as:

$$\bar{E}_t(x,y,z) = \bar{e}(x,y)\boxed{\left(A^+e^{-j\beta_z z}+A^-e^{+j\beta_z z}\right)} \xrightarrow{} \text{leads to } V(z) \qquad (4.6a),(15)$$

and

$$\bar{H}_t(x,y,z) = \bar{h}(x,y)\boxed{\left(A^+e^{-j\beta_z z}-A^-e^{+j\beta_z z}\right)} \xrightarrow{} \text{leads to } I(z) \qquad (4.6b),(16)$$

Where we've just found in (7) & (14) that for a particular mode (m,n)

$$\bar{h}(x,y) = \frac{1}{Z_W}\,\hat{z}\times\bar{e}(x,y) \qquad (4.7),(17)$$

and $Z_W = Z_{TE}$ or Z_{TM} as appropriate.

From (15) & (16) we can quickly identify scalar equations that are directly proportional to transverse $\bar{E}$ & $\bar{H}$ and dependent on z only. In particular, we'll define from (15)

$$\bar{E}_t(x,y,z) = \bar{e}(x,y)\cdot\frac{V(z)}{C_1} \qquad (18)$$

while from (16) $$\bar{H}_t(x,y,z) = \bar{h}(x,y)\cdot\frac{I(z)}{C_2} \qquad (19)$$

s.t. $$V(z) = V^+e^{-j\beta_z z} + V^-e^{+j\beta_z z} \qquad (4.8a),(20)$$

and $$I(z) = I^+e^{-j\beta_z z} - I^-e^{+j\beta_z z} \qquad (4.8b),(21)$$

Comparing (20), (18) & (15) we find that $\frac{V^+}{C_1} = A^+$ & $\frac{V^-}{C_1} = A^-$

or $$C_1 = \frac{V^+}{A^+} = \frac{V^-}{A^-} \quad (22)$$ Similarly, comparing (21), (19), & (16)

$$C_2 = \frac{I^+}{A^+} = \frac{I^-}{A^-} \quad (23).$$

Further, the ratio $\frac{V^+}{I^+}$ is to be associated with the wave impedance. From (22) & (23),

$$\underline{\frac{C_1}{C_2}} = \frac{\frac{V^+}{A^+}}{\frac{I^+}{A^+}} = \frac{V^+}{I^+} \equiv \underline{Z_W} \quad (24).$$

Similarly from (22) & (23):

$$\frac{C_1}{C_2} = \frac{V^-}{I^-} = Z_W \quad (25).$$

Hence, the equivalent TL model eqns. for the hollow metallic waveguide are from (20) & (21) using (24) & (25)

$$\boxed{\begin{aligned} V(z) &= V^+ e^{-j\beta_z z} + V^- e^{+j\beta_z z} & (26) \\ I(z) &= \frac{V^+}{Z_W} e^{-j\beta_z z} - \frac{V^-}{Z_W} e^{+j\beta_z z} & (27) \end{aligned}}$$

These TL equations are very similar to those that model TEM modes. Primary differences are:

- The TL is intrinsically dispersive because β_z is a fct. of freq
- The TL impedance is intrinsically frequency dependent
- This TL models a single propagating mode (m, n).
- The equivalent V(z) & I(z) are proportional to the

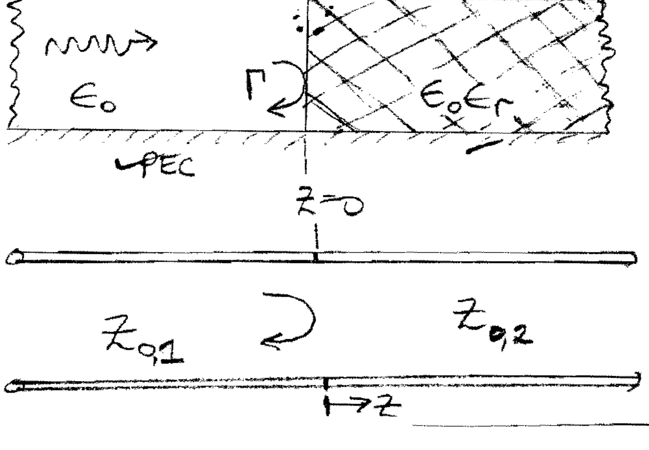
amplitudes of the transverse $\bar{E}$ & $\bar{H}$, respectively. This is in contrast to the TL model of TEM waves where V is proportional to a line integral of $\bar{E}$ in a plane transverse to z and I is related to a closed line integral of $\bar{H}$ in the same plane enclosing one of the conductors.

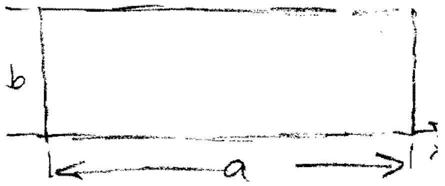
Application of TL Model

The utility of the TL model for hollow metallic waveguides is that we can use it to simplify the analysis of certain waveguide problems. An example of such a problem is the jct. of two homogeneously filled waveguides, as in the next example.

(Text example 4.2)

Example: A WR-90 waveguide is half filled w/ Rexolite, as shown below. Determine the electric field reflection coeff. Γ for the TE_{10} mode at z = 0 using an equivalent TL model for a freq. of 8 GHz. Plot over X band.





For WR-90 wvgd, $a = 2.286$ cm (0.9") and $b = 1.016$ cm (0.4")

For the TE$_{m0}$ modes,

$$f_{c,m0} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\frac{m\pi}{a} = \frac{m c_0}{2a\sqrt{\epsilon_r}}$$

The two lowest-order modes in the two waveguides are:

	air ($\epsilon_r=1$)	Rexolite ($\epsilon_r=2.54$)
m=1:	$f_{c,10} = 6.557$ GHz	$f_{c,10} = 4.11$ GHz
m=2:	$f_{c,20} = 13.11$ GHz	$f_{c,20} = 8.23$ GHz

At 8 GHz, only TE$_{10}$ mode propagates in both waveguides.

For TE^z modes, the wave impedance is given in (5) to be

$$Z_{TE} = \frac{\omega\mu}{\beta_z}$$

For these equivalent TLs

$$Z_{0,1} = \frac{\omega\mu_0}{\beta_{z,10}} = \frac{\omega\mu_0}{\sqrt{\beta_0^2 - (\frac{\pi}{a})^2}} = 657.6\,\Omega$$

and

$$Z_{0,2} = \frac{\omega\mu_0}{\sqrt{\beta_0^2\epsilon_r - (\frac{\pi}{a})^2}} = 275.6\,\Omega$$

smaller than because "farther" from cutoff.

® The electric field ref. coeff. can be computed from the equivalent TL model in the traditional fashion as

$$\underline{\underline{\Gamma}} = \frac{Z_{0,2} - Z_{0,1}}{Z_{0,2} + Z_{0,1}} \underline{\underline{= -0.41}} \text{ @ 8GHz} \qquad (28)$$

(Why is the equivalent to the electric field ref. coeff. in the wvgd?)

Plot of Γ vs. f is shown on next page over the frequency range where single mode prop. in both guides. Very large ref @ ~6.6 GHz because guide "a" near cutoff.

As f increases, thinking of bouncing rays, Γ should approach that of a UPW normally incident on a half space of Rexolite:

$$\Gamma_{TEm} = \frac{\eta_r - 1}{\eta_r + 1} = \frac{\sqrt{\frac{1}{\epsilon_r}} - 1}{\sqrt{\frac{1}{\epsilon_r}} + 1} = -0.229.$$

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In[75]:= ClearAll[є0, μ0, GHz, a, b, βz, m, n, f, єr, Z0TE, Γ, єr1, єr2, freq]

In[76]:= a = 0.9 * 2.54 / 100.  ;
         b = 0.4 * 2.54 / 100.  ;

         є0 = 8.854 * 10^-12  ;
         μ0 = 4. * Pi * 10^-7  ;
         GHz = 10^9  ;

In[81]:= βz[m_, n_, f_, єr_] := Sqrt[(2. * Pi * f)^2 * μ0 * є0 * єr - (m * Pi / a)^2 - (n * Pi / b)^2]
         Z0TE[m_, n_, f_, єr_] := 2. * Pi * f * μ0 / βz[m, n, f, єr]
         Γ[m_, n_, f_, єr1_, єr2_] :=
          (Z0TE[m, n, f, єr2] - Z0TE[m, n, f, єr1]) / (Z0TE[m, n, f, єr2] + Z0TE[m, n, f, єr1])

In[84]:= єr1 = 1.  ;
         єr2 = 2.54  ;
         freq = 8. * GHz ;
         βz[1, 0, freq, єr1]
         βz[1, 0, freq, єr2]
         Z0TE[1, 0, freq, єr1]
         Z0TE[1, 0, freq, єr2]
         Γ[1, 0, freq, єr1, єr2]
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Out[87]= 96.0495 $= \beta_{z,a}$

Out[88]= 229.167 $= \beta_{z,b}$

Out[89]= 657.634 $= Z_{0,a}$

Out[90]= 275.63 $= Z_{0,b}$

Out[91]= -0.40932 $= \Gamma$

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In[100]:=
        Plot[Γ[1, 0, f * GHz, єr1, єr2], {f, 6.558, 8.229},
         AxesLabel → {"f  [GHz]", "Γ"}, PlotStyle → {RGBColor[1, 0, 0]}, PlotRange → All]
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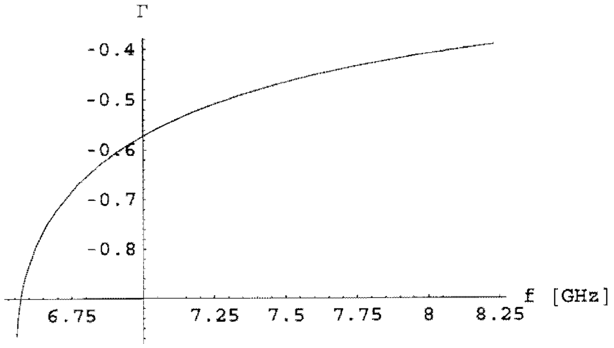

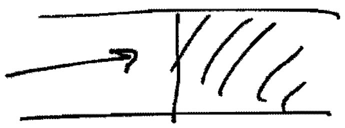

Out[100]=
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- Graphics -
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