Transmission lines are not the only way to guide EM signals. There are other structures that can serve this purpose, and have advantages (and some disads.) compared to TEs. Examples of such structures include hollow metallic pipes, dielectric fibers, and dielectric slabs. We'll focus the remainder of the semester on hollow metallic pipes and cavities.

We'll focus first on rectangular metallic waveguides and resonant cavities for characterizing materials.

![Diagram of rectangular waveguide]

Types of EM Modes in Waveguides

\[ \nabla \times E = -j \omega \mu H \quad \text{and} \quad \nabla \times H = j \omega \varepsilon E \]

Beginning in the Maxwell curl sense in a homogeneous, sourcefree region and assuming propagation in the +z direction as \( e^{-j \beta_z z} \), the transverse components of \( E \) and \( H \) can be expressed as

\[ H_x = \frac{j}{\beta_z} \left( \omega \varepsilon \frac{\partial E_y}{\partial y} - \beta_z \frac{\partial H_y}{\partial x} \right) \quad (1) \]

\[ H_y = \frac{-j}{\beta_z} \left( \omega \varepsilon \frac{\partial E_x}{\partial x} + \beta_z \frac{\partial H_x}{\partial y} \right) \quad (2) \]
What is a "mode"? All is a self-consistent set of fields that satisfy Maxwell's equations and the boundary conditions of the given geometry.

\[
E_x = \frac{-1}{\beta_c^2} \left( \beta_c^2 \frac{\partial E_z}{\partial x} + \omega \mu \frac{\partial H_z}{\partial y} \right)
\]  
(3)

\[
E_y = \frac{j}{\beta_c^2} \left( -\beta_c^2 \frac{\partial E_z}{\partial y} + \omega \mu \frac{\partial H_z}{\partial x} \right)
\]  
(4)

where \( \beta_c^2 \equiv \beta^2 - \beta_z^2 \) and \( \beta^2 = \omega^2 \mu \varepsilon \).

Can define three types of modes in metallic waveguides based on these results:

1. **TE\(_2\)** modes. With \( E_z = 0 \) and \( H_z \neq 0 \) in (1)-(4), then we see that all transverse components of \( E \) and \( H \) can be determined once \( H_z \) is known.

2. **TM\(_2\)** modes. With \( H_z = 0 \) and \( E_z \neq 0 \) in (1)-(4), then we see that all transverse components of \( E \) and \( H \) can be determined once \( E_z \) is known.

3. **TEM\(_2\)** mode. With \( E_z = H_z = 0 \) and \( \beta_c^2 = 0 \) then a mode with strictly transverse \( E \) and \( H \) may be possible. This mode requires at least two conductors to prop. Not possible in hollow metallic waveguides.

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**TE\(_2\)** modes

With \( E_z = 0 \) and \( H_z \neq 0 \), then the transverse field equations (1)-(4) become
Once we have determined $H_2$, one can determine all the transverse field components. How to determine $H_2$?

1. Find governing equation. Wave equation.

2. Solve governing equation: can do this if interior region is a separate geometry.

\[
H_x = -j \frac{\rho_2}{\beta_2} \frac{\partial H_2}{\partial x} \quad (6), \quad E_x = -j \omega \mu_0 \frac{\partial H_2}{\partial y} \quad (8)
\]

\[
H_y = -j \frac{\rho_2}{\beta_2} \frac{\partial H_2}{\partial y} \quad (7), \quad E_y = \frac{j \omega \mu_0}{\beta_2} \frac{\partial H_2}{\partial x} \quad (9)
\]

Once we've analytically determined $H_2$, subject to the appropriate boundary condition, we can use $(6)-(9)$ to solve for all the other field components. Nice!

How do we determine $H_2$? Need to form the governing sign for this field. Begin w/ Maxwell's curl signs in a source-free, homogeneous region.

\[
\nabla \times \vec{E} = -j \omega \mu \vec{H} \quad (10)
\]

\[
\nabla \times \vec{H} = j \omega \varepsilon \vec{E} \quad (11)
\]

Taking curl of $(1): \nabla \times \nabla \times \vec{E} = -j \omega \mu \nabla \times \vec{H} \quad (12)$

Sub $(11) \rightarrow (12)$ gives:

\[
\nabla \times \nabla \times \vec{E} = -j \omega \mu (j \omega \varepsilon \vec{E})
\]

\[
= \omega^2 \mu \varepsilon \vec{E} \quad (13)
\]

Employing the vector identity:

\[
\nabla \times \nabla \times \vec{E} = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E}
\]

and noting that in a source free, homogeneous space

\[
\nabla \cdot \vec{E} = 0 \Rightarrow \nabla \cdot \vec{E} = 0 \quad \text{or} \quad \nabla \cdot \vec{E} = 0
\]

Then $(13)$ becomes

\[
\nabla^2 \vec{E} + \beta^2 \vec{E} = 0 \quad (14)
\]

Similarly, can show

\[
\nabla^2 \vec{H} + \beta^2 \vec{H} = 0 \quad (15)
\]
Eqs. (14) and (15) are the Helmholtz equations for $E \times H$. These are wave equations. For our work here, (15) is the governing eqn. we will solve for $H_2$. This equation is actually a compact way of expressing three scalar equations:

$$\nabla^2 H_x + \beta^2 H_x = 0$$
$$\nabla^2 H_y + \beta^2 H_y = 0$$
$$\nabla^2 H_z + \beta^2 H_z = 0$$

We'll use (18) as the governing eqn. which we need to solve for $H_2$. Because $\mu_1 = \mu_0 + \epsilon \mu \epsilon$ has been assumed, we define

$$H_2(x, y, z) = h_2(x, y) e^{-j\beta_z z}$$

Sub. (19) into (18) gives

$$\nabla^2 h_2(x, y) e^{-j\beta_z z} + \beta^2 h_2(x, y) e^{-j\beta_z z} = 0.$$  

The first term in (20) expands out to

$$(\nabla^2 + \beta^2) h_2(x, y) e^{-j\beta_z z}$$

$$= \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) h_2(x, y) e^{-j\beta_z z} + \beta^2 h_2(x, y) e^{-j\beta_z z}$$

$$= e^{-j\beta_z z} \nabla^2 h_2(x, y) + h_2(x, y) \left( -\beta^2 \right) e^{-j\beta_z z}$$

$$= e^{-j\beta_z z} \left[ \nabla^2 h_2(x, y) - \beta^2 h_2(x, y) \right]$$

where we have defined $\nabla^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$.
Sub (23) into (22) and dividing by $f_0$ gives

\[ h(x) = f(x) \cdot g(y) \]

where $g^2 = \beta^2 - p_2^2$ as defined in (5). Equation (22) becomes the equation of the nucleon wave function.

Sub (23) into (21) gives

\[ \triangle^2 h(x,y) + (\beta^2 - p_2^2) h(x,y) e^{-j\phi/2} = 0 \]

because the solution of the nucleon wave function is a separable geometry, we may reason to suggest that separation of variables can be applied to solve (23).
\[
\frac{1}{f} \frac{d^2f}{dx^2} + \frac{1}{f} \frac{d^2f}{dy^2} = -\beta_e^2 \tag{24}
\]

Looking at the "big picture," the two terms on the LHS must sum to a constant value \( f(x,y) \) because the RHS is a constant (i.e., not a function of space, though it is a function of \( f(x,y) \)).

If this is true then:

So, because the 2\textsuperscript{nd} term in (24) is not a function of \( x \) then the 1\textsuperscript{st} term in (24) we also not a function of \( x \)!

Now, this doesn't mean \( f \) isn't a function of \( x \); this only means that after taking 2 \( x \)-derivatives of \( f \) the derivative \( d^2f/dx^2 \) will be constant. We'll identify this constant as \(-\beta_x^2\) so that:

\[
\frac{1}{f} \frac{d^2f}{dx^2} = -\beta_x^2 \quad \text{or} \quad \frac{d^2f}{dx^2} + \beta_x^2 f = 0 \tag{25}
\]

A similar argument for the 2\textsuperscript{nd} term in (24) leads to

\[
\frac{d^2f}{dy^2} + \beta_y^2 f = 0. \tag{26}
\]

Further, Sub. (23) \& (26) into (24) leads to

\[-\beta_x^2 - \beta_y^2 = -\beta_e^2 \]

\[\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta_e^2 \equiv (\omega^2 \mu) \tag{27}\]

Equation (27) is referred to as the dispersion relation for linearly propagating electromagnetic waves in the medium.
Eqs. (28) and (29) are 2nd-order O.D.E.'s and have known solutions. The form we'll find useful is:

\[ f(x) = A \cos \beta x + B \sin \beta x \]  \hspace{1cm} (28)

\[ g(y) = C \cos \beta y + D \sin \beta y \]  \hspace{1cm} (29)

Sub. (28) \( f(x) \) to (23):

\[ h_2(x,y) = \left( A \cos \beta x + B \sin \beta x \right) \left( C \cos \beta y + D \sin \beta y \right) \]  \hspace{1cm} (30)

\[ h_2(x,y,z) = h_2(x,y) e^{-j \beta z} \]

That's it! We now have an analytical form for \( h_2 \).

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Apply B.C.'s For TE\(_2\) modes

To evaluate the constants \( A - D \) in (30) we need to apply the b.c.'s of the problem. With TEC walls, the b.c.'s are \( \text{Ex} = 0 \) on all four walls.

For TE\(_2\) modes this means:

- \( \text{Ex} = 0 \) for \( 0 \leq x \leq a \) at \( y = 0 \) & \( b \), \( \forall z \) \hspace{1cm} (31)
- \( \text{Ey} = 0 \) for \( 0 \leq y \leq b \) at \( x = 0 \) & \( a \), \( \forall z \) \hspace{1cm} (32)

Defining

\[ E_x(x,y,z) = E_x(x,y) e^{-j \beta z} \]  \hspace{1cm} (33)
... We'll use (8) (9) to determine $E_x, E_y$ from $h_x$:

\[ E_x(x,y) = \frac{\omega x}{\beta_k^2} \frac{\partial h_x}{\partial y} = \frac{\omega x}{\beta_k^2} \left[ A \cos \beta_k x + B \sin \beta_k x \right]. \quad (34) \]

... and

\[ E_y(x,y) = \frac{\omega x}{\beta_k^2} \frac{\partial h_x}{\partial x} = \frac{\omega x}{\beta_k^2} \left[ -A \beta_k \sin \beta_k x + B \beta_k \cos \beta_k x \right]. \quad (35) \]

... Enforce b.c. (31) using (34):

... \* $A + y = 0 \Rightarrow E_x(x,0) = -\frac{\omega x}{\beta_k^2} (A \cos \beta_k x + B \sin \beta_k x) \cdot DP_y = 0$

... Requires $D = 0$.

... \* $A + y = b, D = 0 \Rightarrow E_x(x,b) = -\frac{\omega x}{\beta_k^2} \left[ A \cos \beta_k x + B \sin \beta_k x \right] (-C \beta_k \sin \beta_k b) = 0$

... For a non-trivial solution ($A = B = 0, or C = 0$) then

\[ \beta_k b = m \pi \quad n = 0, 1, 2, \ldots \]

... or

\[ \beta_k = \frac{m \pi}{b}, \quad n = 0, 1, 2, \ldots \quad (36) \]

... Enforce b.c. (32) using (35):

... \* $A + x = 0 \Rightarrow E_y(0,y) = \frac{\omega x}{\beta_k^2} B \beta_k \left[ C \cos \left( \frac{m \pi y}{b} \right) \right]$

... \w/ $D = 0, \beta_k = \frac{m \pi}{b}$

... Requires $B = 0$.
\[ E_\ell (a, y) = \frac{i \omega}{P_\ell^2} \left[ -A \beta_x \sin(\beta_x a) \right] \left[ C \cos \left( \frac{mn}{b} y \right) \right] = 0 \]

Requires for nontrivial solutions that

\[ \beta_x a = m\pi \]

or

\[ \beta_x = \frac{m\pi}{a} \quad \text{if} \quad m = 0, 1, 2, \ldots \quad (37) \]

Altering the results from applying (35) into (38)

\[ H_2(x, y, z) = A_{mn} \cos \left( \frac{m\pi}{a} x \right) \cos \left( \frac{mn}{b} y \right) e^{-i\kappa z} \quad (38) \]

\( A_{mn} \) is a single constant that arose by “absorbing” the product of constants \( A \cdot C \) in (30) into a single one. The subscript indicates it's possible this constant will be different depending on its specific source & the indices \( m \) and \( n \).

Lastly, from the dispersion relation (27):

\[ \beta_x^2 - \beta_y^2 - \beta_z^2 = \beta^2 \quad (36) \]

Sub (36) & (37) we find that

\[ \beta_{x, m n}^2 = \beta^2 - \left( \frac{mn}{a} \right)^2 - \left( \frac{mn}{b} \right)^2 \quad \text{if} \quad m, n = 0, 1, 2, \ldots \quad (39) \]

\( m = n \neq 0 \)
The other field components are $E_z = 0$ and

$$E_x = \frac{i\omega\mu}{\varepsilon} \frac{mr}{b} A_{mn} \cos\left(\frac{mr}{a} x\right) \sin\left(\frac{mr}{b} y\right) e^{-j\beta_z mn z}$$

(40)

$$E_y = \frac{-i\omega\mu}{\varepsilon} \frac{mr}{a} A_{mn} \sin\left(\frac{mr}{a} x\right) \cos\left(\frac{mr}{b} y\right) e^{-j\beta_z mn z}$$

(41)

$$H_x = \frac{jB_{zmn}}{\varepsilon} \frac{mr}{a} A_{mn} \sin\left(\frac{mr}{a} x\right) \cos\left(\frac{mr}{b} y\right) e^{-j\beta_z mn z}$$

(42)

$$H_y = \frac{jB_{zmn}}{\varepsilon} \frac{mr}{b} A_{mn} \cos\left(\frac{mr}{a} x\right) \sin\left(\frac{mr}{b} y\right) e^{-j\beta_z mn z}$$

(43)

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**TM$^2$ Modes**

The derivation of the field solutions for TM$^2$ modes in waveguides directly parallels that for TE$^2$ modes.

For TM$^2$ modes, $E_z \neq 0$ while $H_z = 0$. Applying separation of variables to the wave equation

$$\nabla^2 E_z + \beta^2 E_z = 0$$

yields

$$E_z(x, y, z) = E_z(y, y) e^{-j\beta_z z}$$

$$= (A \cos \beta x + B \sin \beta x)(C \cos \beta_y y + D \sin \beta_y y) e^{-j\beta_z z}$$

(44)
We can apply b.c.'s directly to $E_2(x, y)$. This leads to

$$E_2(x, y, z) = B_{mn} \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) e^{-jkz}$$

$$W, n = 1, 2, \ldots$$

where $B_{mn}$ is given in (39), as for $E_2$ modes. Note in (45) that neither $m$ nor $n$ can equal 0. Also, either case, there would be a trivial solution for the field.

The transverse components can be found from (1)–(4) with $H_3 = 0$ and $E_3$ given in (45). (See remark.)