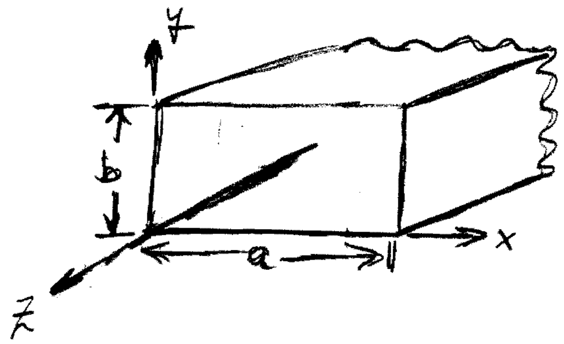
Types of EM modes in Metallic Wvgds. TE^z and TM^z Modal Fields in Rectangular Waveguides.

Transmission lines are not the only way to guide EM signals. There are other structures that can serve this purpose, and have advantages (& some disads.) compared to TLs. Examples of such structures include hollow metallic pipes, dielectric fibers, and dielectric slabs. We'll focus the remainder of the semester on hollow metallic pipes and cavities.

We'll focus first on rectangular metallic waveguide and resonant cavities for characterizing materials.



Types of EM Modes in Waveguides

$\nabla \times \bar{E} = -j\omega\mu \bar{H}$ and $\nabla \times \bar{H} = j\omega\epsilon \bar{E}$

Beginning w/ the Maxwell curl eqns in a homogeneous, sourcefree region and assuming prop. in the $+z$ direction as $e^{-j\beta_z z}$, the transverse components of $\bar{E}$ & $\bar{H}$ can be expressed as

$$H_x = \frac{j}{\beta_c^2}\left(\omega\epsilon \frac{\partial E_z}{\partial y} - \beta_z \frac{\partial H_z}{\partial x}\right) \tag{1}$$

$$H_y = \frac{-j}{\beta_c^2}\left(\omega\epsilon \frac{\partial E_z}{\partial x} + \beta_z \frac{\partial H_z}{\partial y}\right) \tag{2}$$

What is a "mode"? It's a self consistent set of fields that satisfy Maxwell's equations and the boundary conditions of the given geometry.

$$E_x = \frac{-j}{\beta_c^2}\left(\beta_z \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y}\right) \quad (3)$$

$$E_y = \frac{j}{\beta_c^2}\left(-\beta_z \frac{\partial E_z}{\partial y} + \omega\mu \frac{\partial H_z}{\partial x}\right) \quad (4)$$

where $\beta_c^2 \equiv \beta^2 - \beta_z^2$ and $\beta^2 = \omega^2 \mu \epsilon$. (5)

Can define three types of modes in metallic wvgds based on these results:

1. $\underline{TE^z \text{ modes}}$. With $E_z = 0$ and $H_z \neq 0$ in (1) – (4) then we see that all transverse components of $\bar{E}$ & $\bar{H}$ can be determined once H_z is known.

2. $\underline{TM^z \text{ modes}}$. With $H_z = 0$ and $E_z \neq 0$ in (1) – (4) then we see that all transverse components of $\bar{E}$ & $\bar{H}$ can be determined once E_z is known.

3. $\underline{TEM^z \text{ mode}}$. With $E_z = H_z = 0$ & $\beta_c^2 = 0$ then a mode w/ strictly transverse $\bar{E}$ & $\bar{H}$ may be possible. This mode requires at least two conductors to prop. Not possible in hollow metallic wvgds.

TE^z modes

With $E_z = 0$ and $H_z \neq 0$, then the transverse field equations (1) – (4) become

Once we have determined H_z, one can determine all the transverse field components. How to determine H_z?
1. Find governing equation: wave equation
2. Solve governing eqn: can do this if interior region is a separable geometry.

$$H_x = \frac{-j\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial x} \quad (6), \qquad E_x = \frac{-j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial y} \quad (8)$$

$$H_y = \frac{-j\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial y} \quad (7), \qquad E_y = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x} \quad (9)$$

Once we've analytically determined H_z, subject to the appropriate boundary conditions, we can use (6) − (9) to solve for all the other field components. Nice!

How do we determine H_z? Need to form the governing eqn for this field. Begin w/ Maxwell's curl eqns in a source-free, homogeneous region

$$\nabla \times \bar{E} = -j\omega\mu \bar{H} \quad (10)$$

$$\nabla \times \bar{H} = j\omega\epsilon \bar{E} \quad (11)$$

Taking curl of (1): $\nabla \times \nabla \times \bar{E} = -j\omega\mu \nabla \times \bar{H}$ (12)

Sub (11) → (12) gives: $\nabla \times \nabla \times \bar{E} = -j\omega\mu(j\omega\epsilon\bar{E})$

$$= \omega^2\mu\epsilon\bar{E} \quad (13)$$

Employing the vector i.d. $\nabla \times \nabla \times \bar{E} = \nabla(\nabla\cdot\bar{E}) - \nabla^2 \bar{E}$ and noting that in a source free, homogeneous space

$$\nabla\cdot\bar{D} = 0 \Rightarrow \epsilon\nabla\cdot\bar{E} = 0 \text{ or } \nabla\cdot\bar{E} = 0$$

then (13) becomes

$$\nabla^2\bar{E} + \beta^2\bar{E} = 0 \quad (14)$$

Similarly, can show

$$\nabla^2\bar{H} + \beta^2\bar{H} = 0 \quad (15)$$

Eqns (14) and (15) are the Helmholtz equations for $\bar{E}$ & $\bar{H}$. These are wave equations. For our wvgd prob, (15) is the governing eqn. we will solve for H_z. This equation is actually a compact way of expressing three scalar equations:

$$\nabla^2 H_x + \beta^2 H_x = 0 \tag{16}$$
$$\nabla^2 H_y + \beta^2 H_y = 0 \tag{17}$$
$$\nabla^2 H_z + \beta^2 H_z = 0 \tag{18}$$

We'll use (18) as the governing eqn. which we need to solve for H_z. Because prop. in +z has been assumed, we'll define

$$H_z(x,y,z) = h_z(x,y)\, e^{-j\beta_z z} \tag{19}$$

Sub. (19) → (18) gives

$$\nabla^2 h_z(x,y)\, e^{-j\beta_z z} + \beta^2 h_z(x,y)\, e^{-j\beta_z z} = 0. \tag{20}$$

The first term in (20) expands out to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) h_z(x,y)\, e^{-j\beta_z z}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) h_z(x,y)\, e^{-j\beta_z z} + \frac{\partial^2}{\partial z^2} h_z(x,y)\, e^{-j\beta_z z}.$$

$$= e^{-j\beta_z z}\, \nabla_t^2 h_z(x,y) + h_z(x,y) \left(-\beta_z^2 \right) e^{-j\beta_z z}$$

$$= e^{-j\beta_z z} \left[\nabla_t^2 h_z(x,y) - \beta_z^2 h_z(x,y) \right] \tag{21}$$

where we've defined $\nabla_t^2 \equiv \nabla^2 - \frac{\partial^2}{\partial z^2}$

What does "separable" mean? It means that inside this waveguide, the boundary conditions lie on entire coordinate surfaces of a separable coordinate system. In particular, planes in the Cartesian coordinate systems.

Sub (21) → (20) gives (Is the exterior region a separable geometry?)

$$e^{-j\beta_z z}\,\nabla_t^2 h_z(x,y) + (\beta^2-\beta_z^2)\,h_z(x,y)\,e^{-j\beta_z z} = 0$$

because $e^{-j\beta_z z} \neq 0 \Rightarrow$

$$\nabla_t^2 h_z(x,y) + \beta_c^2\, h_z(x,y) = 0 \qquad (22)$$

where $\beta_c^2 \equiv \beta^2 - \beta_z^2$ as defined in (5). Eqn (22) is known as a reduced wave eqn. Once we solve (22) analytically for h_z then H_z is known because of (19).

Because the exterior of the rectangular waveguide is a separable geometry, we have reason to suspect that separation of variables can be applied to solve the pde (22).

By the separation of variables, we assume a product sol'n to h_z in the form

$$h_z(x,y) = f(x)\cdot g(y) \qquad (23)$$

Notice that each of the unknown fcts f & g are dependent on one coordinate variable only.

Sub (23) → (22) and dividing by $f \cdot g$ gives

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} = -\beta_c^2 \qquad (24)$$

Looking at the "big picture," the two terms on the LHS must sum to a constant value $\forall\, x, y$ because the RHS is a constant (i.e., not a fct. of space, though it is a fct. of freq.).

So, because of this (the fact that) the 2^{nd} term in (24) is not a fct. of x then the 1^{st} term in (24) is also not a fct. of x! Now, this doesn't mean f isn't a fct of x; this only means that after taking 2 x-derivatives of f & then dividing by f will be constant. We'll identify this constant as $-\beta_x^2$ so that:

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\beta_x^2 \quad \text{or} \quad \frac{d^2 f}{dx^2} + \beta_x^2 f = 0 \qquad (25).$$

A similar argument for the 2^{nd} term in (24) leads to

$$\frac{d^2 g}{dy^2} + \beta_y^2 g = 0. \qquad (26)$$

Further, sub. (25) & (26) into (24) leads to

$$-\beta_x^2 - \beta_y^2 = -\beta_c^2$$

or

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 (= \omega^2 \mu \epsilon) \qquad (27)$$

Eqn (27) is referred to as the <u>dispersion relation</u> for EM waves propagating down the waveguide.

Eqns (25) & (26) are 2^{nd}-order O.D.E.'s and have known solutions. The form we'll find useful is:

$$f(x) = A\cos\beta_x x + B\sin\beta_x x \qquad (28)$$
$$g(y) = C\cos\beta_y y + D\sin\beta_y y \qquad (29)$$

Sub. (28) & (29) → (23):

$$h_z(x,y) = \left(A\cos\beta_x x + B\sin\beta_x x\right)\left(C\cos\beta_y y + D\sin\beta_y y\right) \qquad (30)$$

and $H_z(x,y,z) = h_z(x,y)\,e^{-j\beta_z z}$

That's it! We now have an analytical sol'n for H_z.

Apply B.C.'s For TE^z modes

To evaluate the constants A–D in (30) we need to apply the b.c.'s of the problem. With PEC walls, the b.c.'s are $E_{tan} = 0$ on all four walls.

For TE^z modes this means

- $E_x = 0$ for $0 \le x \le a$ at $y = 0$ & b, $\forall z$ (31)
- $E_y = 0$ for $0 \le y \le b$ at $x = 0$ & a, $\forall z$ (32)

Defining $E_{x,y}(x,y,z) = e_{x,y}(x,y)\,e^{-j\beta_z z}$ (33)

Will use (8) & (9) to determine e_x & e_y from h_z:

$$e_x(x,y) = \frac{-j\omega\mu}{\beta_c^2}\frac{\partial h_z}{\partial y} = \frac{-j\omega\mu}{\beta_c^2}\left[A\cos\beta_x x + B\sin\beta_x x\right]\cdot\left[-C\beta_y\sin\beta_y y + D\beta_y\cos\beta_y y\right] \qquad (34)$$

and

$$e_y(x,y) = \frac{j\omega\mu}{\beta_c^2}\frac{\partial h_z}{\partial x} = \frac{j\omega\mu}{\beta_c^2}\left[-A\beta_x\sin\beta_x x + B\beta_x\cos\beta_x x\right]\cdot\left[C\cos\beta_y y + D\sin\beta_y y\right] \qquad (35)$$

Enforce b.c. (31) using (34):

- At $y=0 \Rightarrow e_x(x,0) = -\frac{j\omega\mu}{\beta_c^2}\left(A\cos\beta_x x + B\sin\beta_x x\right)\cdot D\beta_y = 0$
  Requires $D=0$.

- At $y=b$ & $D=0 \Rightarrow$
  $$e_x(x,b) = -\frac{j\omega\mu}{\beta_c^2}\left(A\cos\beta_x x + B\sin\beta_x x\right)\left(-C\beta_y\sin\beta_y b\right) = 0$$

  For a non-trivial ~~solution~~ ($A=B=0$, or $C=0$) then
  $$\beta_y b = n\pi \qquad n=0,1,2,\cdots$$
  or
  $$\beta_y = \frac{n\pi}{b} \qquad n=0,1,2,\cdots \qquad (36)$$

Enforce b.c. (32) using (35):

- At $x=0 \Rightarrow e_y(0,y) = \frac{j\omega\mu}{\beta_c^2}B\beta_x\left[C\cos\left(\frac{n\pi y}{b}\right)\right]$
  w/ $D=0$ & $\beta_y = \frac{n\pi}{b}$

  Requires $B=0$

• At $x=a$ ($\because B=D=0,\ \beta_y=\frac{n\pi}{b}$) $\Rightarrow$

$$e_y(a,y)=\frac{j\omega\mu}{\beta_c^2}\left[-A\beta_x\sin(\beta_x a)\right]\left[C\cos\left(\frac{n\pi}{b}y\right)\right]=0$$

Requires for nontrivial solutions that

$$\beta_x a = m\pi$$

or

$$\underline{\underline{\beta_x=\frac{m\pi}{a}}}\quad,\ m=0,1,2,\cdots \tag{37}$$

Incorporating the results from applying b.c.'s into (30)

$$H_z(x,y,z)=A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}\right)e^{-j\beta_{z,mn}z} \tag{38}$$

Amn is a single constant that arose by "absorbing" the product of constants $A\cdot C$ in (30) into a single one. The subscript indicates it's possible this constant will be different depending on the specific source & the indices m and n.

Lastly, from the dispersion relation (27):

$$\beta_z^2=\beta^2-\beta_x^2-\beta_y^2$$

Sub. (36) & (37) we find that

$$\beta_{z,mn}^2=\beta^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2\quad m,n=0,1,2,\cdots \tag{39}$$

$(m=n\neq 0)$

The other field components are $E_z = 0$ and

$$E_x = \frac{j\omega\mu}{\beta_c^2} \cdot \frac{n\pi}{b} A_{mn} \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{-j\beta_{z,mn} z} \tag{40}$$

$$E_y = \frac{-j\omega\mu}{\beta_c^2} \frac{m\pi}{a} A_{mn} \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) e^{-j\beta_{z,mn} z} \tag{41}$$

$$H_x = \frac{j\beta_{z,mn}}{\beta_c^2} \frac{m\pi}{a} A_{mn} \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) e^{-j\beta_{z,mn} z} \tag{42}$$

$$H_y = \frac{j\beta_{z,mn}}{\beta_c^2} \frac{n\pi}{b} A_{mn} \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{-j\beta_{z,mn} z} \tag{43}$$

TM^z Modes

The derivation of the field solutions for TM^z modes in waveguides directly parallels that for TE^z modes.

For TM^z modes, $E_z \neq 0$ while $H_z = 0$ Applying separation of variables to the wave eqn

$$\nabla^2 E_z + \beta^2 E_z = 0$$

gives $E_z(x,y,z) = e_z(x,y) e^{-j\beta_z z}$

$$= (A \cos \beta_x x + B \sin \beta_x x)(C \cos \beta_y y + D \sin \beta_y y) e^{-j\beta_z z} \tag{44}$$

We can apply b.c.'s directly to $e_z(x,y)$. This leads to

$$E_z(x,y,z) = B_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_{z,mn}z} \qquad (45)$$

$$m,n = 1,2,\cdots$$

where $\beta_{z,mn}$ is given in (39), as for TE^z modes. Note in (45) that neither m nor n can equal 0. In either case, there would be a trivial solution for the fields

The transverse field components can be found from (1)–(4) with $H_z = 0$ and E_z given in (45). (See hwork.)