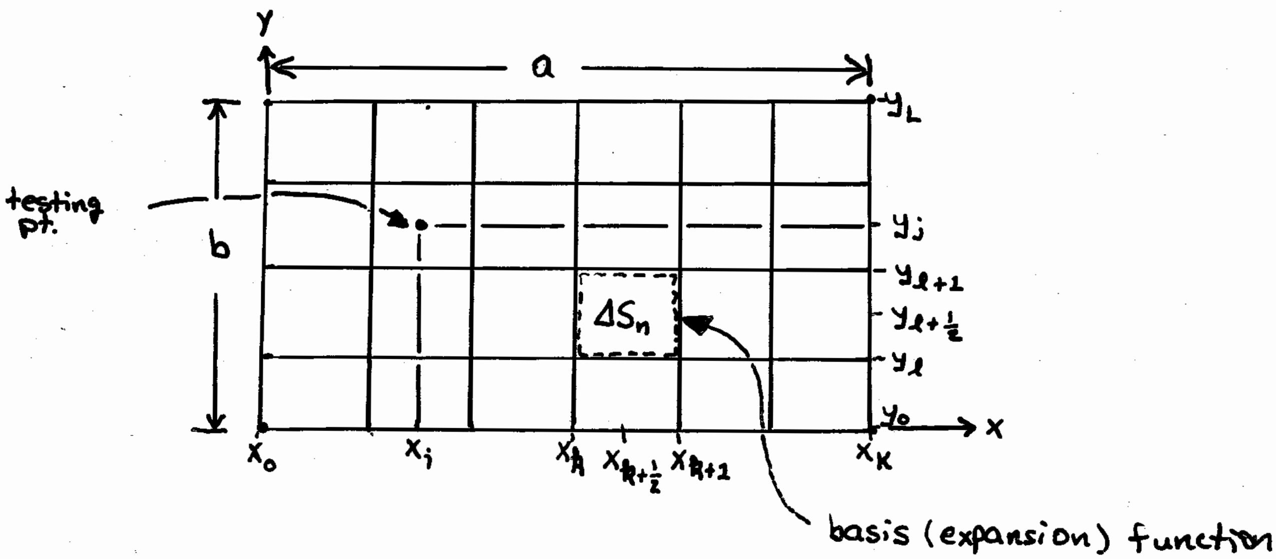
Moment Method Solution to the Static Plate Example

The integral equation for the surface charge density, ρ_s, on the plate having a prescribed potential V impressed upon it was previously derived as

$$V = \frac{1}{4\pi\epsilon} \int_{\text{plate}} \frac{\rho_s(x', y')}{|\bar{r} - \bar{r}'|} \, dx' dy'$$

$$\bar{r}, \bar{r}' \in \{\text{pts. on plate}\}$$

Since there are no derivatives appearing in this equation, we can accurately expand the charge with the simplest expansion (basis) functions $\rightarrow$ <u>2-D pulses</u>.



Expand surface charge - let

$$\rho_s(x,y) \approx \sum_{k=1}^{K} \sum_{\ell=1}^{L} \alpha_{k\ell} \underbrace{P_k(x; x_k, x_{k+1})}_{\text{pulse in } x} \underbrace{P_\ell(y; y_\ell, y_{\ell+1})}_{\text{pulse in } y}$$

In Harrington's notation –

$$\rho_s(x,y) \approx \sum_{n=1}^{N} \alpha_n f_n \quad \text{where} \quad f_n = \begin{cases} 1 & \text{on } \Delta S_n \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad \underline{N = K \cdot L}$$

The <u>testing</u> functions will be chosen as <u>delta functions</u> located in the centers of <u>each</u> cell of the charge discretization ("<u>point matching</u>") –

$$w_{ij} = \delta(x - x_i)\,\delta(y - y_j) \qquad \text{where} \quad \begin{aligned} i &= 1, \cdots, K \\ j &= 1, \cdots, L \end{aligned}$$

In Harrington's notation,

$$w_m = \delta(x - x_m)\,\delta(y - y_m) = \delta(\bar{r} - \bar{r}_m)$$

where $\bar{r}_m$ = position of centroid of the m^{th} cell w/ $m = 1, \cdots, M$

To form a system of eqns. w/ an equal number of equations and unknowns, choose

$$\underline{\underline{M = K \cdot L}} \qquad (\text{test at centroid of } \underline{\text{every}} \text{ cell})$$

From our previous discussions on the MM, the resulting set of algebraic equations, after substituting for the basis fct. expansion & then testing, can be written in matrix form –

$$[l_{mn}][\alpha_n] = [g_m]$$

where

$$l_{mn} = \langle w_m, \mathcal{L}\{f_n\}\rangle$$
$$g_m = \langle w_m, V\rangle$$
$$\alpha_n = \text{unknowns}$$

Now let's evaluate the matrix elements l_{mn} →

$$\mathcal{L}\{f_n\} = \frac{1}{4\pi\epsilon}\iint_{plate} \frac{f_n}{|\bar{r}-\bar{r}'|}\,dx'dy' = \frac{1}{4\pi\epsilon}\int_{y_l}^{y_{l+1}}\int_{x_k}^{x_{k+1}} \frac{1}{[(x-x')^2+(y-y')^2]^{1/2}}\,dx'dy'$$

then,

$$\langle w_m, \mathcal{L}\{f_n\}\rangle = \frac{1}{4\pi\epsilon}\iint_{plate}\delta(\bar{r}-\bar{r}_m)\int_{y_l}^{y_{l+1}}\int_{x_k}^{x_{k+1}} \frac{1}{[(x-x')^2+(y-y')^2]^{1/2}}\,dx'dy'\,dxdy$$

$$\Rightarrow l_{mn} = \frac{1}{4\pi\epsilon}\int_{y_l}^{y_{l+1}}\int_{x_k}^{x_{k+1}} \frac{1}{[(x_i-x')^2+(y_j-y')^2]^{1/2}}\,dx'dy' \qquad (1)$$

This integral is evaluated for each element in $[l_{mn}]$. As m & n change, so do $\{k,l\}$ and $\{i,j\}$ giving different values to the integral.

In most mm problems, l_{mn} cannot be evaluated in closed form. However, for this static plate problem we can evaluate l_{mn} →

For $m \neq n$ –

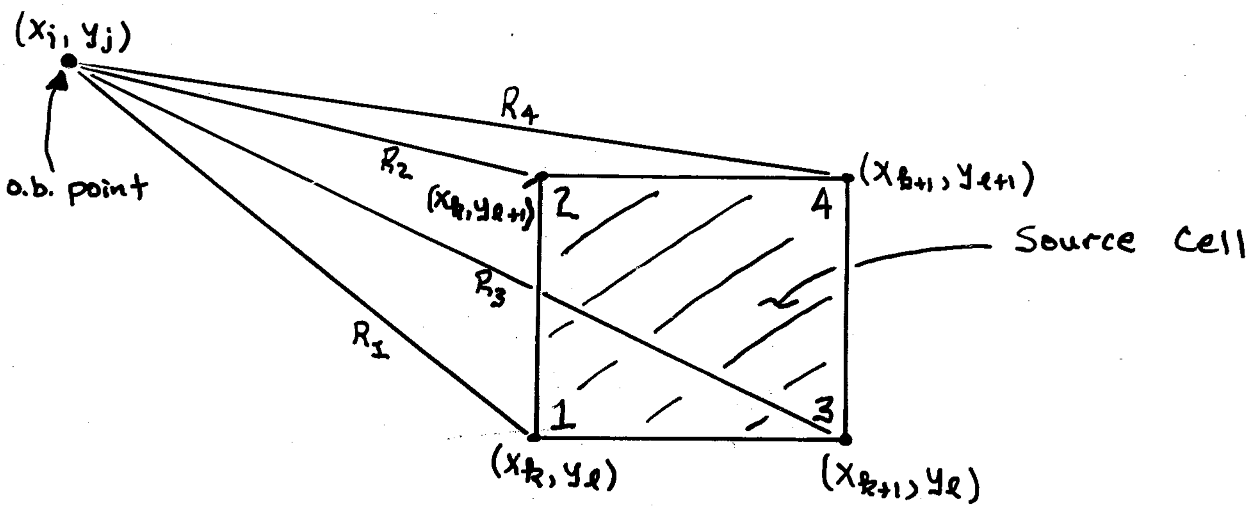
$$l_{mn} = \frac{1}{4\pi\epsilon_0}\left\{ (x_i - x_k)\,\ln\left[\frac{y_j - y_l + R_1}{y_j - y_{l+1} + R_2}\right] + (y_j - y_l)\,\ln\left[\frac{x_i - x_k + R_1}{x_i - x_{k+1} + R_3}\right] \right.$$

$$\left. + (x_i - x_{k+1})\,\ln\left[\frac{y_j - y_{l+1} + R_4}{y_j - y_l + R_3}\right] + (y_j - y_{l+1})\,\ln\left[\frac{x_i - x_{k+1} + R_4}{x_i - x_k + R_2}\right] \right\} \quad (2)$$

where –

$$R_1 = \left[(x_i - x_k)^2 + (y_j - y_l)^2\right]^{1/2}, \quad R_3 = \left[(x_i - x_{k+1})^2 + (y_j - y_l)^2\right]^{1/2}$$

$$R_2 = \left[(x_i - x_k)^2 + (y_j - y_{l+1})^2\right]^{1/2}, \quad R_4 = \left[(x_i - x_{k+1})^2 + (y_j - y_{l+1})^2\right]^{1/2}$$



Although we have computed l_{mn} in closed form analytically, most times this is not possible. One common method of reducing the computational expense in the "matrix fill" in $\langle w_m, \mathcal{L}\{f_n\}\rangle = l_{mn}$ (a 4-D integral !) is to use what is called a <u>one-point approximation</u>.

This one-pt. approx. means that the integrand in ① for example can be approximated by a <u>constant value over the entire cell</u> – using the value computed @ the centroid.

From ①, this gives the approximate expression

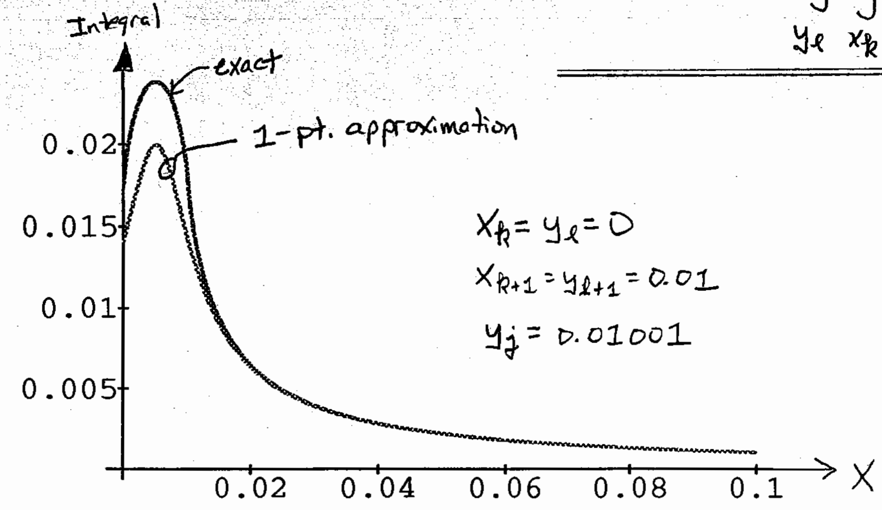
$$(m \neq n) \qquad l_{mn} \approx \frac{\Delta S_n}{4\pi\epsilon} \frac{1}{\left[(x_i - x_{k+\frac{1}{2}})^2 + (y_j - y_{l+\frac{1}{2}})^2\right]^{1/2}} \qquad ③$$

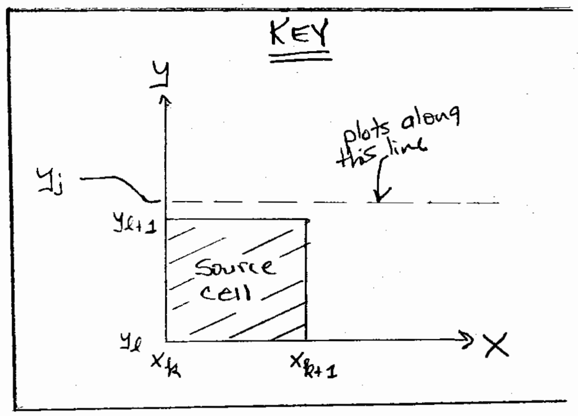
cell centroid locations

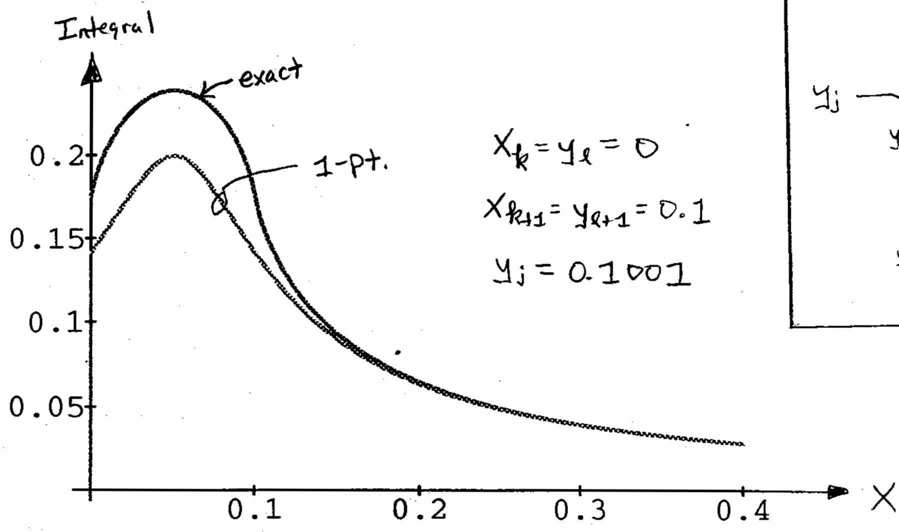
This approximation becomes more and more valid as the o.b. point moves further away from the source cell as seen in the fig on the preceeding page.

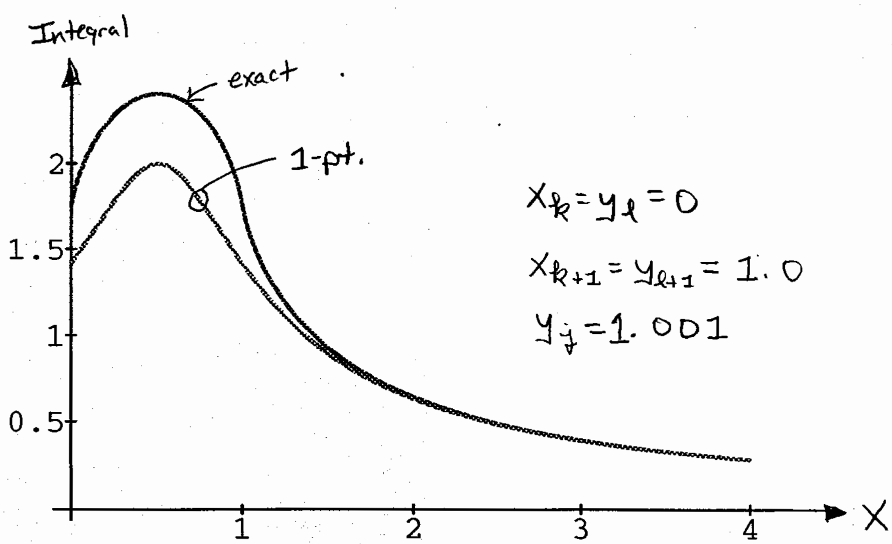
Rule of Thumb – When o.b. pt is 2-3 <u>basis</u> cells away from source cell, this one-pt. approximation is nearly exact !

$$\text{Integral} = \int_{y_\ell}^{y_{\ell+1}} \int_{x_k}^{x_{k+1}} \frac{1}{\sqrt{(x-x')^2+(y_j-y')^2}}\, dx'dy'$$









Using ② or ③ we can evaluate all of the elements in $[l_{mn}]$ except for the $m=n$ elements. That is, ② & ③ are valid for the non-diagonal elements of $[l_{mn}]$.

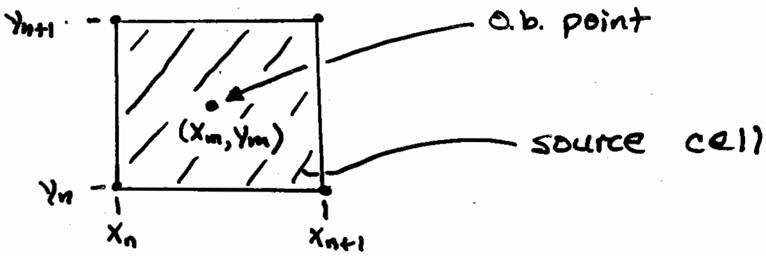
$$\begin{array}{cc} & \xrightarrow{\quad n \quad} \text{(source)} \\ \text{(test)}\ m \big\downarrow & \begin{bmatrix} (1,1) & (1,2) & (1,3) & \cdots & \\ (2,1) & (2,2) & (2,3) & \cdots & \\ & & & & \\ & & \cdots & (N-1,N-1) & (N-1,N) \\ & & \cdots & (N,N-1) & (N,N) \end{bmatrix} \end{array} = [l_{mn}]$$

diagonal elements = self-cell terms

The diagonal (or self-cell) elements are a special difficulty in all mm problems.

For $m=n$:

$$l_{mm} = \frac{1}{4\pi\epsilon} \int_{y_n}^{y_{n+1}} \int_{x_n}^{x_{n+1}} \frac{1}{\left[(x_m - x')^2 + (y_m - y')^2\right]^{1/2}} \, dx' dy' \qquad ④$$

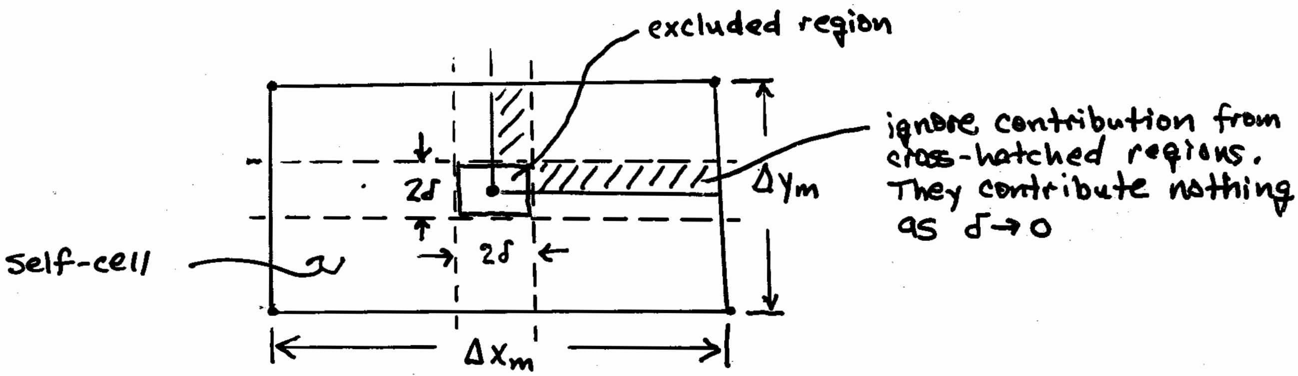


Due to the singularity in the integrand of ④ at the centroid, this integral must be evaluated in the Cauchy Principal Value sense.

A P.V. integral is evaluated by excluding a small region (surface, volume) about the singularity. A limiting value of the integral is obtained as the size of this excluded region vanishes.

By symmetry we can write ④ as –

$$l_{mn} = \frac{4}{4\pi\epsilon} \lim_{\delta\to 0} \left\{ \int_{y_m+\delta}^{y_{n+1}} \int_{x_m+\delta}^{x_{n+1}} \frac{1}{\left[(x_m-x')^2 + (y_m-y')^2\right]^{1/2}} dx'dy' \right\}$$



We can evaluate the above integral in closed form using ② since the singular point has been excluded. From ② letting

$$x_n = x_m + \delta$$
$$y_n = y_m + \delta$$

$$l_{mm} = \frac{1}{\pi\epsilon} \lim_{\delta\to 0} \left\{ -\delta \ln\left[\frac{-\delta+\sqrt{\delta^2+\delta^2}}{y_m - y_{n+1} + R_2}\right] + (-\delta)\ln\left[\frac{-\delta+\sqrt{\delta^2+\delta^2}}{x_m - x_{n+1} + R_3}\right] + (x_m - x_{n+1})\ln\left[\frac{y_m - y_{n+1} + R_4}{-\delta + R_3}\right] + (y_m - y_{n+1})\ln\left[\frac{x_m - x_{n+1} + R_4}{-\delta + R_2}\right] \right\} \quad ⑤$$

Examine the first two terms in (5). What are their limits as $\delta \to 0$? Both terms can be written in the form

$$\lim_{\delta\to 0}\left\{-\frac{\ln\left[\frac{(\sqrt{2}-1)\delta}{K}\right]}{\delta^{-1}}\right\} = \lim_{\delta\to 0}\left\{-\frac{\frac{1}{\delta}}{-\delta^{-2}}\right\}$$

K = constant $\neq f(\delta)$

L'Hospital's Rule

$$= \lim_{\delta\to 0}\{\delta\} \longrightarrow \underline{\underline{0}}$$

Therefore, the first two terms in (5) vanish as $\delta \to 0$.

We're now left w/

$$l_{mm} = \frac{1}{\pi\epsilon}\left\{(x_m - x_{m+1})\ln\left[\frac{y_m - y_{m+1} + R_4}{|x_m - x_{m+1}|}\right] + (y_m - y_{m+1})\ln\left[\frac{x_m - x_{m+1} + R_4}{|y_m - y_{m+1}|}\right]\right\}$$

$$= \frac{1}{\pi\epsilon}\left\{\frac{\Delta x_m}{2}\ln\left[\frac{\frac{\Delta y_m}{2} + \left[\left(\frac{\Delta x_m}{2}\right)^2 + \left(\frac{\Delta y_m}{2}\right)^2\right]^{1/2}}{\frac{\Delta x_m}{2}}\right] + \right.$$

$$\left.\frac{\Delta y_m}{2}\ln\left[\frac{\frac{\Delta x_m}{2} + \left[\left(\frac{\Delta x_m}{2}\right)^2 + \left(\frac{\Delta y_m}{2}\right)^2\right]^{1/2}}{\frac{\Delta y_m}{2}}\right]\right\} \quad (6)$$

Exact expression for all self-cell terms (diagonal elements of $[l_{mn}]$).

Special case – if $\Delta x_m = \Delta y_m = \Delta_m$ (square cells) $\Rightarrow$

$$l_{mm} = \frac{1}{\pi\epsilon}\left\{ \frac{\Delta_m}{2} \ln\left[1+\sqrt{2}\right] + \frac{\Delta_m}{2} \ln\left[1+\sqrt{2}\right] \right\}$$

or $$l_{mm} = \frac{\Delta_m}{\pi\epsilon} \ln\left[1+\sqrt{2}\right] \qquad (7)$$

Agrees w/ Harrington, eqn. (2-31)

The <u>next step</u> in the mm solution to the quasi-static plate problem is to evaluate the excitation vector $[g_m]$.

$$g_m = \langle w_m, V \rangle$$

$$= V \int_{plate} \delta(\bar{r} - \bar{r}_m)\, dx\, dy$$

$$\Rightarrow \quad \underline{g_m = V} \qquad (8)$$

<u>Summarizing</u>

For the matrix eqn $[l_{mn}][\alpha_n] = [g_m]$

- Eqn (2) is the expression for the l_{mn} elements w/ $m \neq n$. Eqn (6) valid for $m = n$ elements
- Eqn (3) is an approx. l_{mn} expression for $m \neq n$ using one-pt. approx.
- Eqn (8) is expression for g_m elements

Questions to Ponder → Using this MM formulation (& code)

(1) Could we analyze a peforated plate? A circular plate?

(2) Could we analyze a 2-plate problem? 3 plates?

(3) How about a dielectric filling inside a 2-plate cap. problem?

(4) Could we analyze a dielectric coating around a single plate?