# Method of Moments

The integral eqn. we wish to solve (quasi-static)

$$V = \int_{s'} \frac{\rho_s(\bar{r}')}{4\pi\epsilon} \frac{1}{|\bar{r}-\bar{r}'|} \, ds' \qquad ①$$

w/ $\bar{r}$ & $\bar{r}' \in \{\text{pts. on plate}\}$

will be approached from a numerical procedure called the Method of Moments (MM)

! The basic idea & central modus operandi for all MM solutions is —

① Expand the unknown, in this case $\rho_s$, in a set of functions which have a known form but unknown amplitude

② Form a matrix equation

③ Solve this system of equations for the amplitudes

④ calculate desired observables. For example in this case $\Phi_e$ anywhere.

This four step procedure is universally valid for most, if not all, MM solutions. Some problems will present unique challenges (thin wires, dielectrics, thin coatings, periodic geometries, etc...) which need to be addressed --→ however, the numerical procedure is little changed.

In operator form, the integral equation ① can be written as

$$\mathcal{L}\{f\} = g \qquad ②$$

where $\underline{g}$ is a <u>Known</u> function & $\underline{f}$ is <u>unknown</u>.

Objective: solve for $f$ as $f = \mathcal{L}^{-1}\{g\}$

The following seven steps outline the mm sol'n of ① →

(i) Expand $f$ in a set of basis functions, $f_n$, which is usually finite. Therefore the $\{f_n\}$ representation of $f$ will then be approximate, i.e.,

$$f \approx \sum_{n=1}^{N} \underbrace{\alpha_n}_{\substack{\text{amplitude} \\ \text{(maybe complex)}}} \underbrace{f_n}_{\substack{\text{Known function} \\ \text{(of space)}}} \qquad ③$$

(ii) Substitute this representation in ③ into the original eqn. ② giving

$$\mathcal{L}\left\{\sum_{n=1}^{N} \alpha_n f_n\right\} = g$$

Key step!

$$\Rightarrow \sum_{n=1}^{N} \alpha_n \mathcal{L}\{f_n\} = g$$

for $\mathcal{L}$ = linear operator

(iii) Determine an appropriate inner product. For example –

$$\langle f, g \rangle = \int f\, g \, dx$$

(iv) Select a set of testing functions ($w_m$) in the domain of $\mathcal{L}$.

(v) Evaluate the inner product of $\mathcal{L}\{f\} = g$ with each testing function $w_m$. This forms a system of equations.

$$\langle w_m, \mathcal{L}\{f\} \rangle = \langle w_m, g \rangle \qquad m = 1, \dots, M$$

total # of testing fcts.

or,
$$\left\langle w_m, \sum_{n=1}^{N} \alpha_n \mathcal{L}\{f_n\} \right\rangle = \langle w_m, g \rangle$$

$$\Rightarrow \sum_{n=1}^{N} \alpha_n \langle w_m, \mathcal{L}\{f_n\} \rangle = \langle w_m, g \rangle \qquad m = 1, \dots, M$$

fct. of $m$ & $n$

testing　　basis expansion

(vi) Solve the resulting set of linear algebraic eqns. for the unknowns $\alpha_n$. For square matrices this is easily done.

Let $M = N$, then

$$\underbrace{\sum_{n=1}^{N} \alpha_n \langle w_m, \mathcal{L}\{f_n\} \rangle}_{\text{square matrix}} = \langle w_m, g \rangle \qquad m = 1, \cdots, \underset{!!}{N}$$

(vii) Solve for unknown coefficients $\alpha_n$. The resulting numerical approximation for the unknown function is then

$$f \approx \sum_{n=1}^{N} \alpha_n f_n$$

both are known.

We can also repeat the steps to this derivation but use matrix notation –

$$\sum_{n=1}^{N} \alpha_n \langle w_m, \mathcal{L}\{f_n\} \rangle = \langle w_m, g \rangle$$

can be written as

$$\sum_{n=1}^{N} l_{mn} \alpha_n = g_m$$

or

$$[l_{mn}][\alpha_n] = [g_m]$$

matrix

vectors

where

$$[l_{mn}] = \begin{bmatrix} \langle w_1, \mathcal{L}\{f_1\}\rangle & \langle w_1, \mathcal{L}\{f_2\}\rangle & \cdots & \langle w_1, \mathcal{L}\{f_N\}\rangle \\ \vdots & & & \\ \langle w_N, \mathcal{L}\{f_1\}\rangle & \langle w_N, \mathcal{L}\{f_2\}\rangle & \cdots & \langle w_N, \mathcal{L}\{f_N\}\rangle \end{bmatrix}$$

N×N matrix

inner product of $N^{th}$ testing fct w/ $\mathcal{L}$ operating on expansion fct. # 2.

$$[\alpha_n] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} \quad \& \quad [g_m] = \begin{bmatrix} \langle w_1, g\rangle \\ \langle w_2, g\rangle \\ \vdots \\ \langle w_N, g\rangle \end{bmatrix}$$

N×1 (vector) N×1 (vector)

If the matrix $[l_{mn}]$ is nonsingular, then $[l_{mn}]^{-1}$ exists and

$$[\alpha_n] = [l_{mn}]^{-1} [g_m]$$

where $[l_{mn}]^{-1}$ is the approximate numerical inverse of the operator $\mathcal{L}$.

# Guidelines when Implementing MM.

The success (or failure) of the mm rests largely on the proper selection of the basis and testing functions. Some things to keep in mind are:

(1) $f_n$ & $w_m$ should be linearly independent combinations of functions

(2) The basis fcts. $f_n$ should be capable of representing $f$ fairly well. (Need to understand the problem beforehand!)

(3) $\langle w_m, g \rangle$ should depend on independent properties of $g$. For example, spread out the testing functions over the body.

(4) $l_{mn}$ should be easily evaluated! (efficient code)

(5) $[l_{mn}]$ must be a well-conditioned matrix

# Basis & Testing Functions

Basis functions (expansion fcts) can be broadly divided into two general types:

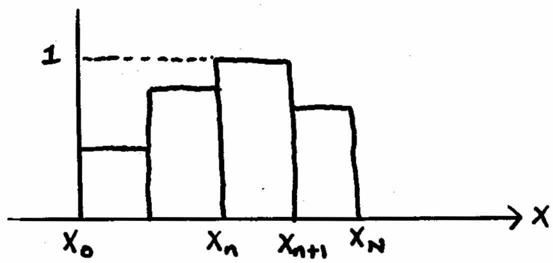
(I) Entire domain – support over the whole domain

(II) Sub-domain – finite support on the domain

Examples of sub-domain basis fcts (in 1-D) include:

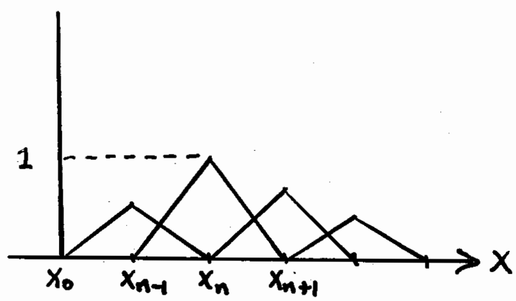
(1) Pulse function : $P_n(x; x_n, x_{n+1})$ ← unit amplitude

not differentiable in the normal sense




(2) Triangle function : $T_n(x; x_{n-1}, x_n, x_{n+1})$ ← unit amplitude

once differentiable (in x) in the normal sense




(3) Sinusoidal triangles, splines, etc...

These functions can also be used as testing functions. When the basis and testing functions have the same form (both are pulse fcts, for example), the process is called Gaylerkin's Method.

Another widely used testing fct which, in certain circumstances, gives accurate results is the Dirac delta function:

$$w_m = \delta(\bar{r} - \bar{r}_m)$$

testing @ the pt. $\bar{r}_m$.

This testing method is also called point-matching since

$$l_{mn} = \langle w_m, \mathcal{L}\{f_n\} \rangle$$

is evaluated only at a discrete number of o.b. pts. If this type of testing fct. is applicable, it can greatly reduce the computational expense when filling the matrix $[l_{mn}]$.

This technique should be avoided (or used w/ great trepidation) if $\mathcal{L}\{f_n\}$ is a symbolic fct., i.e., has for example Dirac delta fcts. contained within it.