

Linear Spaces

(Ref. Harrington, Appendix A)

An element in a linear space is a definable mathematical quantity - some basic unit. Examples include (1) scalars, (2) functions, (3) vectors & (4) matrices.

A space (S) is a collection of these elements considered as a whole.

A linear space exists if each pair of elements f & g in S can be combined in a process called addition to yield another element

$$h = f + g \quad \text{in } S$$

and each element f in S can be combined with a scalar α by a process called multiplication (by a scalar) to yield another element αf in S such that :

- | | |
|--|-------------------------------|
| (a) $f + g = g + f$ | addition commutes |
| (b) $f + (g + h) = (f + g) + h$ | associative |
| (c) $\alpha(f + g) = \alpha f + \alpha g$ | mult. distributive w.r.t. add |
| (d) $(\alpha + \beta)f = \alpha f + \beta f$ | associative |
| (e) $\alpha(\beta f) = (\alpha\beta)f$ | commutes |
| (f) $1 \cdot f = f$ | define 1 |
| (g) $0 \cdot f = 0$ | define 0 |

An inner product of two ^{elements} ~~vectors~~ f & g in S is a scalar denoted as $\langle f, g \rangle$ such that -

$$(a) \quad \langle f, g \rangle = \langle g, f \rangle$$

$$(b) \quad \langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

$$(c) \quad \langle f, f^* \rangle > 0 \quad \text{if } f \neq 0 \\ = 0 \quad \text{if } f = 0$$

where α & β are scalars and ' $*$ ' denotes complex conjugate.

Considering two nonempty spaces A w/ elements a_i and B w/ elements b_i , a mapping M is a rule whereby to each element a_i of A there corresponds an element b_i of B :

$$b = M(a)$$

A mapping is linear if:

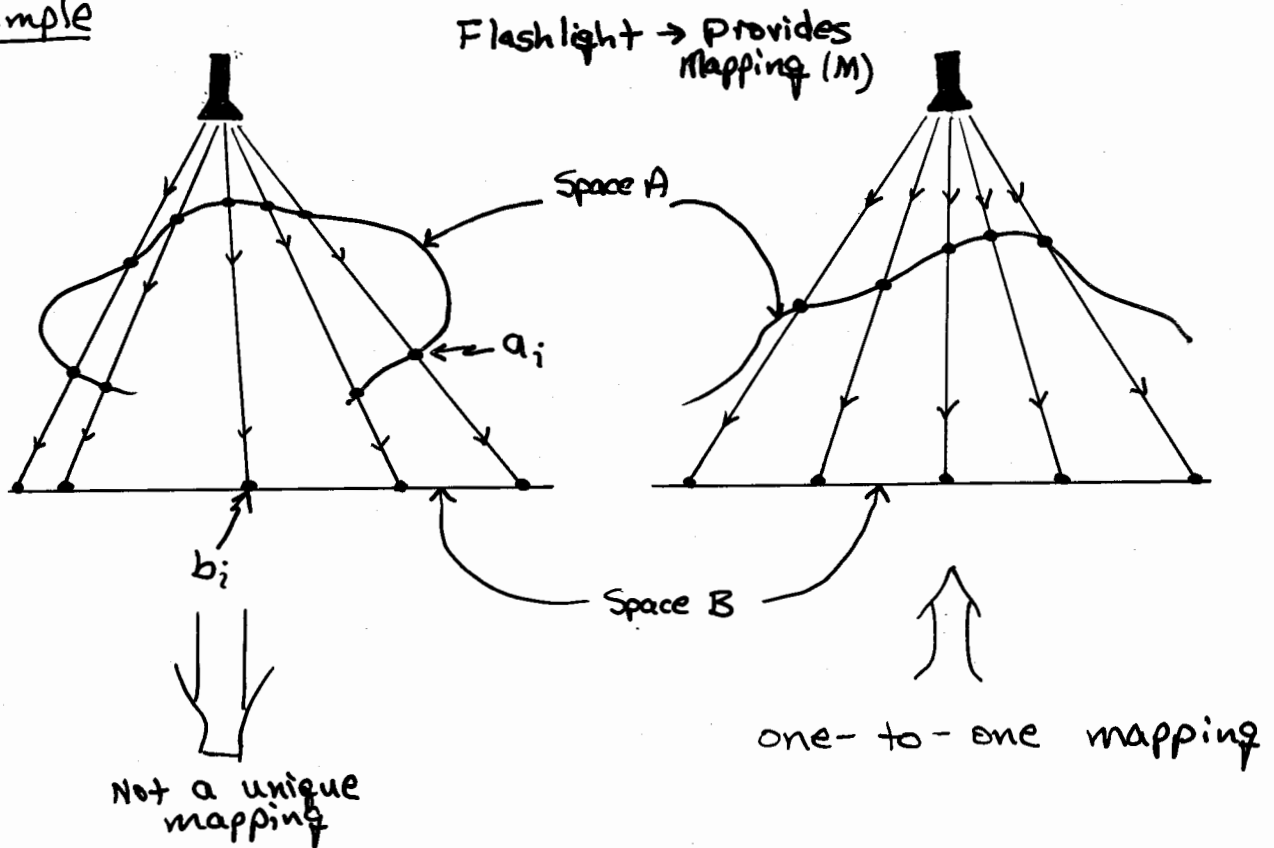
$$M(a_1 + a_2) = M(a_1) + M(a_2)$$

$$M(\alpha a) = \alpha M(a)$$

where α is a scalar.

If there is a one-to-one correspondence in the mapping, then the inverse mapping from B to A is defined as

$$M^{-1}(b) = a$$

Example

Examples of mappings include -

(a) functions

(b) functionals

(c) operators : $g = \mathcal{L}\{f\}$ maps from a function space F w/ elements f into a function space G w/ elements g .

The domain of M is the space A on which M operates. The range of M is the space B resulting from the mapping.

The elements of a linear space S are linearly independent if

$$\sum \alpha_i u_i = 0$$

implies $\alpha_i = 0 \quad \forall i$.

A set of linearly independent elements spans S if every element in S can be expressed as

$$f = \sum \alpha_i u_i .$$

The set $\{u_i\}$ forms a basis for S if the α_i are unique.

An example of a linear space :

$$E_3 : u = [u_1, u_2, u_3] \quad \text{Euclidean 3 space.}$$

(ordered set of 3 scalars)

In EM, this space is further refined to a 3-D Maxwellian space since these complex functions of real variables can not just be any functions. They must also satisfy Maxwell's equations.

Examples of inner (scalar) products -

$$E_3 : \langle u, v \rangle = \sum u_i v_i$$

$$L : \langle f, g \rangle = \int f(x) g(x) dx$$

These forms are similar. Makes sense - can think of a function in an infinite-dimensional vector space.