Linear Spaces

(Ref. Harrington, Appendix A)

An <u>element</u> in a linear space is a definable mathematical quantity — some basic unit. Examples include (1) scalars, (2) functions, (3) vectors & (4) matrices.

A <u>space</u> (S) is a collection of these elements considered as a whole.

A <u>linear space</u> exists if each pair of elements f & g in S can be combined in a process called <u>addition</u> to yield another element

$$h = f + g \quad \text{in } S$$

and each element ~~in~~ f in S can be combined with a scalar α by a process called <u>multiplication</u> (by a scalar) to yield another element αf <u>in S</u> such that :

(a)	$f+g = g+f$	addition commutes
(b)	$f+(g+h) = (f+g)+h$	associative
(c)	$\alpha(f+g) = \alpha f + \alpha g$	mult. distributive w.r.t. add
(d)	$(\alpha+\beta) f = \alpha f + \beta f$	associative
(e)	$\alpha(\beta f) = (\alpha\beta) f$	commutes
(f)	$1 \cdot f = f$	define 1
(g)	$0 \cdot f = 0$	define 0

An <u>inner product</u> of two ~~vectors~~ elements f & g in S is a scalar denoted as $\langle f, g \rangle$ such that -

(a) $\langle f, g \rangle = \langle g, f \rangle$

(b) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$

(c) $\langle f, f^* \rangle > 0$ if $f \neq 0$

$= 0$ if $f = 0$

where α & β are scalars and '$*$' denotes complex conjugate.

Considering two nonempty spaces A w/ elements a_i and B w/ elements b_i, a <u>mapping</u> M is a rule whereby to each element a_i of A there corresponds an element b_i of B:

$$b = M(a).$$

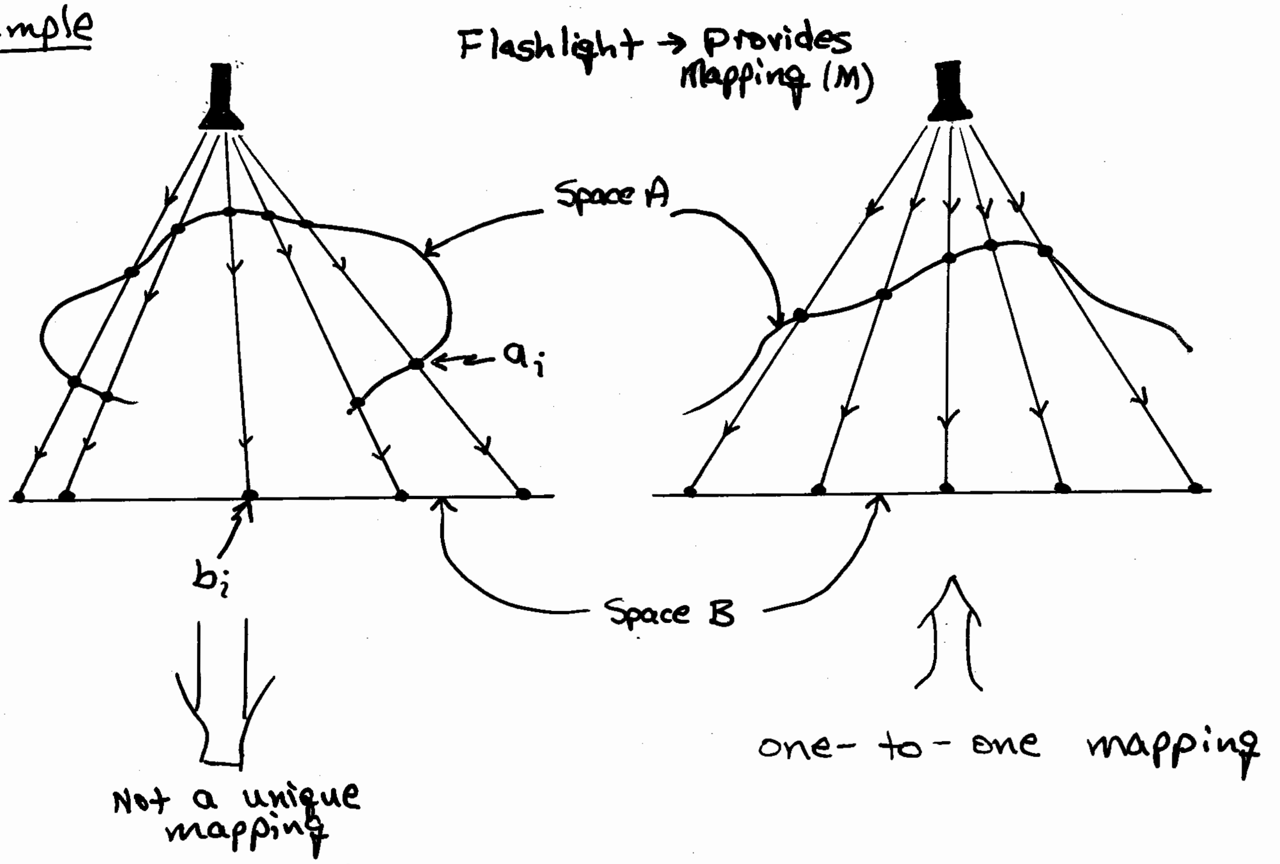
A mapping is <u>linear</u> if:
$$M(a_1 + a_2) = M(a_1) + M(a_2)$$
$$M(\alpha a) = \alpha M(a)$$

where α is a scalar.

<u>If</u> there is a <u>one-to-one</u> <u>correspondence</u> in the mapping, then the inverse mapping from B to A is defined as

$$M^{-1}(b) = a.$$

Example



Examples of mappings include -

(a) functions

(b) functionals

(c) operators : $g = \mathcal{L}\{f\}$ maps from a function space F w/ elements f into a function space G w/ elements g.

The <u>domain</u> of M is the space A on which M operates. The <u>range</u> of M is the space B resulting from the mapping.

The <u>elements</u> of a linear space S (u_i) are <u>linearly independent</u> if

$$\sum \alpha_i u_i = 0$$

implies $\alpha_i = 0 \quad \forall i$.

A set of linearly independent elements <u>spans</u> S if every element in S can be expressed as

$$f = \sum \alpha_i u_i \; .$$

The set $\{u_i\}$ forms a <u>basis</u> for S if the α_i are unique.

An example of a linear space:

$E_3 \quad : \quad u = [u_1, u_2, u_3]$ Euclidean 3 space.
(ordered set of 3 scalars)

In EM, this space is further refined to a 3-D Maxwellian space since these complex functions of real variables can not just be any functions. They must also satisfy Maxwell's equations.

Examples of inner (scalar) products -

E_3 : $\langle u, v \rangle = \Sigma\, u_i v_i$

L : $\langle f, g \rangle = \int f(x)\, g(x)\, dx$

These forms are similar. Makes sense - can thick of a function in an infinte - dimensional vector space.