EPIE and the Quasi-Static Integral Eqn. for a Conducting Plate

From the vector potential method discussed earlier, the magnetic vector potential $\bar{A}$ was found to satisfy the P.D.E.

$$\nabla^2 \bar{A} + k^2 \bar{A} = -\bar{J}$$

having the sol'n : $\bar{A}(\bar{r}) = \bar{J}(\bar{r}') * g(\bar{r}, \bar{r}')$.

We can also derive a P.D.E. for the scalar electric potential, Φ_e.

Starting from Maxwell's equations –

$$\nabla \times \bar{E} = -j\omega\mu \bar{H} \quad \& \quad \nabla \times \bar{H} = j\omega\epsilon \bar{E} + \bar{J} \qquad ①$$

In the usual fashion, define $\mu\bar{H} = \nabla \times \bar{A}$

$$\bar{E} = -j\omega\mu \bar{A} - \nabla \Phi_e \qquad ②$$

Take $\nabla \cdot$ ② $\rightarrow$ $\nabla \cdot \bar{E} = -j\omega\mu \nabla \cdot \bar{A} - \nabla \cdot \nabla \Phi_e$

Now take $\nabla \cdot$ ① $\rightarrow$ $\nabla \cdot \nabla \times \bar{H} = 0 = j\omega\epsilon \nabla \cdot \bar{E} + \nabla \cdot \bar{J}$

Now sub in for $\nabla \cdot \bar{E}$ giving:

$$-j\omega\epsilon \left[-j\omega\mu \nabla \cdot \bar{A} - \nabla \cdot \nabla \Phi_e \right] = \nabla \cdot \bar{J} \qquad ③$$

using the continuity eqn : $\nabla\cdot\bar{J} = -j\omega\rho_e$

& the Lorentz gauge : $\nabla\cdot\bar{A} = -j\omega\epsilon\nabla\Phi_e$,

then ③ can be written as

$$-j\omega\mu(-j\omega\epsilon\,\Phi_e) - \nabla^2\Phi_e = +\frac{\rho_e}{\epsilon}$$

or

$$\underline{\underline{\nabla^2\Phi_e + k^2\Phi_e = -\frac{\rho_e}{\epsilon}}} \qquad ④$$

The scalar electric potential must satisfy a P.D.E. of exactly the same form as each Cartesian component of the vector magnetic potential. The only difference is the forcing fct!

Then by analogy, the solutions to ④ are given as

$$\Phi_e(\bar{r}) = \frac{\rho_e(\bar{r}')}{\epsilon} * g(\bar{r},\bar{r}')$$

or,

$$\Phi_e(\bar{r}) = \int_{V'} \frac{\rho_e(\bar{r}')}{\epsilon}\,\frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}\,dv'$$

source volume

For quasi-~~static~~ problems, i.e., as was shown earlier for the point dipole (for $|kr| \ll 1$), the fields have the same form as the static problem — but they vary harmonically with time.

For this problem, then as $|kr|$ becomes very small →

$$\nabla^2 \Phi_e + \cancelto{0}{k^2 \Phi_e} = -\frac{\rho_e}{\epsilon}$$

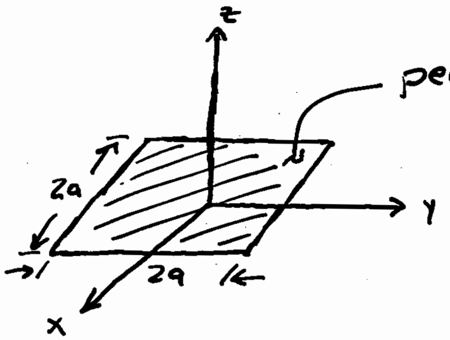
$$\Downarrow$$

$$\Phi_e \approx \int_{V'} \frac{\rho_e(\bar{r}')}{\epsilon} \cdot \frac{1}{4\pi |\bar{r}-\bar{r}'|} \, dv'$$

For <u>static</u> ($\omega = 0$) problems, this solution is <u>exact</u>!

EPIE for Plate

The first problem we're going to numerically solve is a pec plate of infinitesimal thickness, having a prescribed potential impressed on it.



As we've determined, the potential in space due to an arbitrary distribution of charge density is

$$\Phi_e(\bar{r}) = \int_{V'} \frac{\rho_e(\bar{r}')}{\epsilon} \frac{1}{4\pi|\bar{r}-\bar{r}'|} \, dv'$$

For a surface charge density, as we have on a pec surface, the integral reduces to

surface charge density

$$\Phi_e(\bar{r}) = \int_{S'} \frac{\rho_s(\bar{r}')}{\epsilon} \cdot \frac{1}{4\pi|\bar{r}-\bar{r}'|} \, ds' \qquad (5)$$

integrate over a surface.

Remember: our objective is to first find ρ_s on the plate (S'). Once that is known, we can find Φ_e anywhere from (5) (static).

To find this ρ_s numerically, we first will apply the <u>boundary conditions</u> imposed on this structure:

B.C.'s : On the plate $|x| \leq a$, $|y| \leq a$, $z=0$

$$\Phi_e(\bar{r}) = V \qquad \bar{r} \in \{\text{pts. on plate}\}.$$

That's it !

So from ⑤ we obtain an electric potential integral equation (EPIE) for ρ_s as

(Harrington, eqn. (2-11))

$$V = \int_{S'} \frac{\rho_s(\bar{r}')}{\epsilon} \frac{1}{4\pi |\bar{r}-\bar{r}'|} \, ds' \qquad ⑥$$

for $\bar{r}, \bar{r}' \in \{\text{pts. on plate}\}$

Equation ⑥ is a very difficult equation to solve analytically. In fact, a closed form solution is surely impossible to obtain (for ρ_s). The difficulty, of course, is that our unknown of interest is part of the integrand!

We will solve eqn ⑥ numerically using what is typically called the Moment Method (MM).