Green's Functions & the Solutions to $\nabla^2 \bar{L} + k^2 \bar{L} = -\bar{f}$

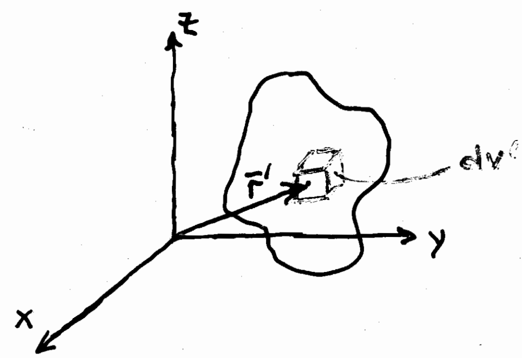
Either of the vector potential or direct method equations shown earlier in Part 2 can be written in the form

$$\nabla^2 \bar{L} + k^2 \bar{L} = -\bar{f}$$

vector forcing function, i.e., the source.

Since these equations are LINEAR in $\bar{L}$, the method of solution will rest on the principle of superposition.

Proceeding then, take a source volume V and break it up into a summation of exceedingly small pieces:



An infinitesimal source element will be described using Dirac delta functions -

$$\delta(\bar{r}-\bar{r}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

where δ is defined (loosely) through the integral

$$\int \delta(\bar{r}-\bar{r}')\, dv' \equiv 1$$

The diff. eqn. for an infinitesimal source element can be written as

$$\nabla^2 L_i + k^2 L_i = -\delta(\bar{r} - \bar{r}') \qquad i = x, y, z$$

primed signifies source coordinate!

As a first step, let $\bar{r}' = 0$ (source @ origin).

Since we're only looking @ a pt. source, most of the volume will be empty. ∴ first find the <u>homogeneous</u> <u>sol'n</u> for $\bar{r} \neq 0$. In that case, the diff. eqn. becomes -

$$\nabla^2 g + k^2 g = 0 \qquad (1)$$

In spherical coordinates, g is only a function of r due to the spherical symmetry of the source. Then,

$$\nabla^2 g = \nabla \cdot (\nabla g) = \nabla \cdot \left(\hat{r} \frac{\partial g}{\partial r}\right)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial g}{\partial r}\right) = \frac{1}{r^2}\left[r^2 \frac{\partial^2 g}{\partial r^2} + 2r \frac{\partial g}{\partial r}\right]$$

$$= \frac{1}{r}\Big[\underbrace{r \frac{\partial^2 g}{\partial r^2} + 2 \frac{\partial g}{\partial r}}_{\text{identical}}\Big]$$

However, consider $\frac{\partial}{\partial r}(rg) = r \frac{\partial g}{\partial r} + g$

then

$$\frac{\partial^2 (rg)}{\partial r^2} = r \frac{\partial^2 g}{\partial r^2} + \frac{\partial g}{\partial r} + \frac{\partial g}{\partial r} = \underbrace{r \frac{\partial^2 g}{\partial r^2} + 2 \frac{\partial g}{\partial r}}_{\text{identical}}$$

$$\therefore \quad \nabla^2 g = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rg)$$ and (1) can be written as

$$\frac{\partial^2}{\partial r^2}(rg) + k^2(rg) = 0$$

There are two solutions to this Diff. eq.

$$rg = Ce^{-jkr} + \underbrace{De^{+jkr}}$$

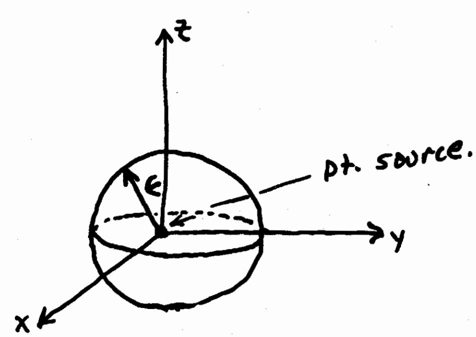
For small losses in the volume, this term $\to \infty$ as $r \to$ large

$\therefore$ set $\underline{D = 0}$

Which leaves the solution -

$$g(r) = C\,\frac{e^{-jkr}}{r} \qquad (2)$$

We must now evaluate the constant C by applying the boundary conditions: Integrate over a small spherical volume centered about the point source.



Integrating $\nabla^2 g + k^2 g = -\delta(r)$

and taking $\lim_{\epsilon \to 0}$ gives —

$$\lim_{\epsilon \to 0} \left\{ \int_0^{\epsilon}\int_0^{2\pi}\int_0^{\pi} \left[\nabla^2 g + k^2 g\right] r^2 \sin\theta \, d\theta \, d\phi \, dr = -\int_V \delta(x)\delta(y)\delta(z)\, dv = -1 \right\}$$

Inside the volume, $g = C \frac{e^{-jkr}}{r} \approx \frac{C}{r}$ since $e^{-jkr} \approx 1$

$\Rightarrow$ $k^2 g \cdot r^2 \approx 0$ inside volume as $\epsilon \to 0$.

also, $\nabla g \approx \nabla \frac{C}{r} = -\hat{r}\frac{1}{r^2}$

Substituting into the integral gives —

$$\lim_{\epsilon \to 0} \left\{ \int_0^{\epsilon}\int_0^{2\pi}\int_0^{\pi} \nabla \cdot \left(-\hat{r}\frac{C}{r^2}\right) r^2 \sin\theta \, d\theta \, d\phi \, dr \right\} = -1$$

Now apply the Divergence Theorem —

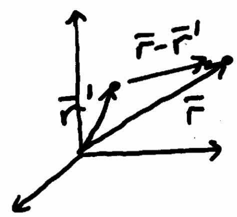
$$\lim_{\epsilon \to 0} \left\{ \int_0^{2\pi}\int_0^{\pi} \left(\frac{C}{r^2}\hat{r}\right) \cdot \hat{r} \, r^2 \sin\theta \, d\theta \, d\phi \right\} = 1$$

or $C(4\pi) = 1$ $\leftarrow$ Notice: no ϵ!

Integrable Singularity in E_3.

which gives $$\underline{\underline{g(\bar{r}) = \frac{e^{-jk|\bar{r}|}}{4\pi|\bar{r}|}}} \tag{3}$$

Now suppose that instead of the point source located at the origin it is displaced by some vector $\bar{r}'$ as



Then if all space is homogeneous, we can simply translate the origin of our new coord. system as

$$\bar{r} \to \bar{r}-\bar{r}' \quad \& \quad |\bar{r}| \to |\bar{r}-\bar{r}'| \qquad \text{in (3)}$$

giving

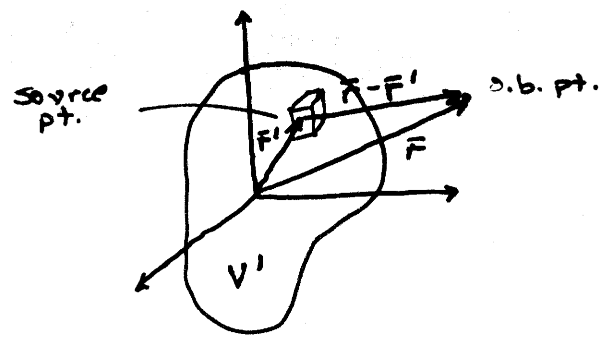
$$\underline{\underline{g(\bar{r},\bar{r}') = g(|\bar{r}-\bar{r}'|) = \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}}} \tag{4}$$

This function $g(\bar{r},\bar{r}')$ is called the Green's function. It is the unit impulse response of the system (a homogeneous, infinite, 3-D space) to a source impulse.

However, our objective is to find solutions to

$$\nabla^2 \bar{L} + k^2 \bar{L} = -\bar{f} \qquad (\text{e.g., } \nabla^2 \bar{A} + k^2 \bar{A} = -\bar{J})$$

We can accomplish this using the impulse response by sub-dividing the source volume into tiny pieces. Using superposition, we the add up the responses to each piece –



$$dL_i = \underbrace{g(|\bar{r}-\bar{r}'|)}_{\text{unit impulse response}} \; \underset{\text{Amplitude}}{f_i} \; \underbrace{dx'dy'dz'}_{\text{small volume}}$$

$i = x, y, z$

or

$$L_i = \int_{V'} f_i(\bar{r}')\, g(|\bar{r}-\bar{r}'|)\, dv'$$

notice amplitude is primed!

Generalizing this last expression –

$$\underline{\underline{\bar{L}(\bar{r}) = \int_{V'} \bar{f}(\bar{r}')\, g(|\bar{r}-\bar{r}'|)\, dv'}} \qquad \leftarrow \text{a convolution}$$

operator

or

$$\bar{L}(\bar{r}) = \bar{f}(\bar{r}') * g(|\bar{r}-\bar{r}'|)$$

we have <u>inverted</u> the operator $\nabla^2 + k^2$

Our two original diff. eqns. earlier were -

Potential method - $\nabla^2 \bar{A} + k^2 \bar{A} = -\bar{J}$

Direct method - $\nabla^2 \bar{E} + k^2 \bar{E} = -\left[\frac{\nabla\nabla\cdot\bar{J} + k^2\bar{J}}{j\omega\epsilon} \right]$

which then have the solutions -

Potential method : $\bar{A}(\bar{r}) = \int_{V'} \bar{J}(\bar{r}')\, g(|\bar{r}-\bar{r}'|)\,dv' = \bar{J}(\bar{r}') * g(|\bar{r}-\bar{r}'|)$

$\Rightarrow$

$$\bar{E}(\bar{r}) = \frac{\nabla\nabla\cdot\bar{A}(\bar{r}) + k^2\bar{A}(\bar{r})}{j\omega\epsilon}$$

$$\bar{H}(\bar{r}) = \nabla\times\bar{A}(\bar{r})$$

Direct method : $\bar{E}(\bar{r}) = \int_{V'} \frac{\nabla'\nabla'\cdot\bar{J}(\bar{r}') + k^2\bar{J}(\bar{r}')}{j\omega\epsilon}\, g(|\bar{r}-\bar{r}'|)\,dv'$ *

$$= \frac{\nabla'\nabla'\cdot\bar{J}(\bar{r}') + k^2\bar{J}(\bar{r}')}{j\omega\epsilon} * g(|\bar{r}-\bar{r}'|)$$

* prime on ∇' means derivative taken on source (primed) coordinates!

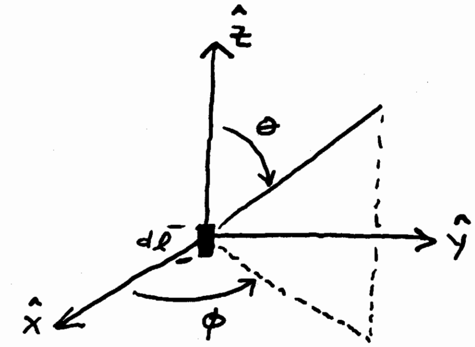
$$\bar{H}(\bar{r}) = -\frac{1}{jk\eta}\nabla\times\bar{E}(\bar{r})$$

* Notice the difference between these two forms. In the first case, we're taking derivatives (o.b. coords.) after the convolution. In the other case, the derivatives are taken on the current density (primed coords), then convolution.

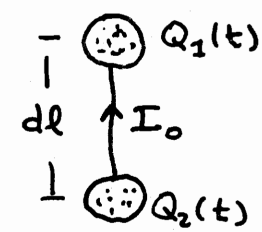
Example – Hertzian Dipole

For a point (or Hertzian) dipole, the source $\bar{J}$ has the form

$$\bar{J} = \hat{z}\underbrace{I_o\, d\ell}_{\text{dipole moment}}\ \delta(\bar{r}-\bar{r}'^{\,\to 0})$$



This current is supported by 2 time-varying accumulations of charge –



$$Q_1(t) = \frac{I_o}{\omega}\sin\omega t$$

$$I(t) = \frac{dQ_1}{dt} = -\frac{dQ_2}{dt} = I_o\cos(\omega t)$$

$$Q_2(t) = -\frac{I_o}{\omega}\sin\omega t$$

From the vector potential method –

$$\bar{A} = \hat{z}\, I_o\, d\ell\, \frac{e^{-jkr}}{4\pi r}$$

$$\therefore\quad \bar{H} = \nabla\times\bar{A} = \hat{\phi}\,\frac{I_o\, d\ell}{4\pi}\sin\theta\left\{\frac{1}{r^2}+\frac{jk}{r}\right\}e^{-jkr} \qquad \hat{\phi} \text{ only}$$

$$\bar{E} = \hat{r}\,\frac{2 I_o\, d\ell\,\eta}{4\pi k}\cos\theta\left\{\frac{k}{r^2}+\frac{1}{jr^3}\right\}e^{-jkr} \qquad \text{no } \hat{\phi}$$

$$+\,\hat{\theta}\,\frac{I_o\, d\ell\,\eta}{4\pi k}\sin\theta\left\{\frac{jk^2}{r}+\frac{k}{r^2}+\frac{1}{jr^3}\right\}e^{-jkr}$$

Consider a special case: Look at the fields very near the dipole $\Rightarrow r \to 0$. Then we can neglect fields which vary as $\frac{1}{r}$ and $\frac{1}{r^2}$ w.r.t. $\frac{1}{r^3}$ variation.

$$\therefore \qquad \underline{\underline{\bar{E} \approx \frac{Q\,d\ell}{4\pi\epsilon r^3}\left[\hat{r}\,2\cos\theta + \hat{\theta}\sin\theta\right]}} \qquad (5)$$

This is precisely the $\bar{E}$ field of a static dipole composed of 2 charge accumulations Q & $-Q$.

But, for our case, Q is a phasor! Q is a fct. of time. So although the form of the solutions are the same for this $\bar{E}$ & the electrostatic $\bar{E}$, the former vary sinusoidally with time.

(*) This $\frac{1}{r^3}$ term in (5) is referred to as the electrostatic term. (*)

When is this "quasi"-static approximation valid for the point dipole? Compare terms in the exact solution (both $\hat{r}$ & $\hat{\theta}$ comps. of $\bar{E}$)

Answer: $\frac{|k|}{r^2} \ll \left|\frac{1}{jr^3}\right|$

$$\Rightarrow \quad |r| \ll \frac{1}{|k|} = \frac{|\lambda|}{2\pi}$$