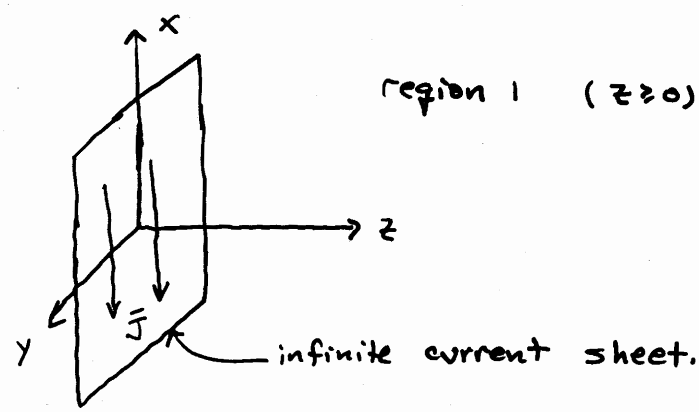
Uniform Plane Waves and Their Sources

One of the simplest "radiation" problems is finding the fields produced by an infinite, planar sheet of impressed surface current -

region 2 ($z \leq 0$) region 1 ($z \geq 0$)



Objective is to solve for the fields produced by this time-harmonic current density -

$$\bar{J} = -\hat{x}\, J_{so}\, \delta(z)$$

Solution methodology -

- Maxwell's eqns.
- Simplify - $\frac{\partial}{\partial x} \to 0$ $\frac{\partial}{\partial y} \to 0$
- form wave eqns (P.D.E.'s)
- Solve - plane wave sol'ns.
- Apply b.c.'s : (1) radiation condition
  (2) jump discontinuity in $\bar{H}$

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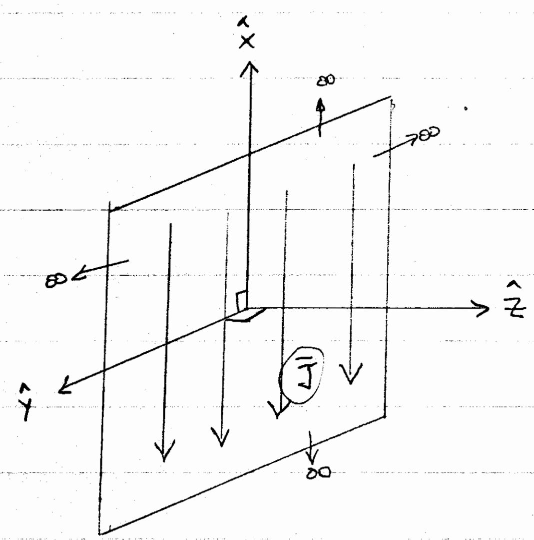
Perhaps one of the simplest "radiation" problems which can be solved analytically is that of an infinite, planar sheet of "impressed" surface current as shown.



This "impressed" current is assumed to be the source of the radiated fields.

Let $\bar{J} = -\hat{x} J_{s_0} \cos \omega t = -\hat{x} J_{s_0} \, Re\{\cancel{e^{j\omega t}}\}$ suppressed

The objective is to solve for the fields which are produced by this time-harmonic current density

$$\bar{J} = -\hat{x} J_{s_0} \, \delta(z)$$

↑ Dirac delta function

Starting with Maxwell's equations -

$$\nabla \times \bar{E} = -j\omega \bar{B} \quad ; \quad \nabla \times \bar{H} = \bar{J} + j\omega \bar{D}$$

using the constitutive parameters $\bar{D} = \epsilon_0 \bar{E}$, $\bar{B} = \mu_0 \bar{H}$
then

$$\nabla \times \bar{E} = -j\omega\mu_0 \bar{H} \quad ; \quad \nabla \times \bar{H} = \bar{J} + j\omega\epsilon_0 \bar{E} \qquad (1)$$

Away from the source and in free space , $\bar{J} = 0$.
Also, since we have an infinite current sheet we can simplify the problem considerably.

$$\nabla \times \bar{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

There is no variation in either the geometry or the source in the x ; y directions so that $\frac{\partial}{\partial x} \to 0$, $\frac{\partial}{\partial y} \to 0$.

$$\therefore \quad \nabla \times \bar{E} = \hat{x}\left(-\frac{\partial E_y}{\partial z}\right) - \hat{y}\left(-\frac{\partial E_x}{\partial z}\right)$$

Similarly $\nabla \times \bar{H} = \hat{x}\left(-\frac{\partial H_y}{\partial z}\right) - \hat{y}\left(-\frac{\partial H_x}{\partial z}\right)$

Substituting into Eqns. (1) →

$$-\hat{x}\,\frac{\partial E_y}{\partial z} + \hat{y}\,\frac{\partial E_x}{\partial z} = -j\omega\mu_0 \bar{H}$$
$$-\hat{x}\,\frac{\partial H_y}{\partial z} + \hat{y}\,\frac{\partial H_x}{\partial z} = j\omega\epsilon_0 \bar{E}$$

The objective now is to combine these equations in such a manner so that the two resulting scalar equations involve only one unknown function. To do this, take $\frac{\partial}{\partial z}$ of the first eqn. and multiply second by $j\omega\mu_0$ giving –

$$-\hat{x}\,\frac{\partial^2 E_y}{\partial z^2} + \hat{y}\,\frac{\partial^2 E_x}{\partial z^2} = -j\omega\mu_0\left(\hat{x}\,\frac{\partial H_x}{\partial z} + \hat{y}\,\frac{\partial H_y}{\partial z} + \hat{z}\,\frac{\partial H_z}{\partial z}\right)$$

$$-\hat{x}\,j\omega\mu_0\,\frac{\partial H_y}{\partial z} + \hat{y}\,j\omega\mu_0\,\frac{\partial H_x}{\partial z} = -\omega^2\mu_0\epsilon_0\left(\hat{x}\,E_x + \hat{y}\,E_y + \hat{z}\,E_z\right)$$

Substiting from the second equation into the first and equating vector components –

$$-\hat{x}\,\frac{\partial^2 E_y}{\partial z^2} = +\hat{x}\,\omega^2\mu_0\epsilon_0 E_y$$

$$\hat{y}\,\frac{\partial^2 E_x}{\partial z^2} = -\omega^2\mu_0\epsilon_0 E_x$$

or

$$\frac{\partial^2 E_y}{\partial z^2} + k_0^2 E_y = 0$$

$$\text{where } k_0^2 \equiv \omega^2\mu_0\epsilon_0 \qquad (2)$$

$$\frac{\partial^2 E_x}{\partial z^2} + k_0^2 E_x = 0$$

Equations (2) are known as wave equations since their solutions give rise to fields which vary in space (and time) as waves.

Their solutions are of the form -

$$E_y = Ae^{-jk_0 z} + Be^{jk_0 z}$$

$$E_x = Ce^{-jk_0 z} + De^{jk_0 z}$$

in <u>each</u> region 1 and 2. The unknown constants A, B, C & D can be determined by applying the boundary conditions.

First B.C. : $\hat{n} \times (\bar{E}_1 - \bar{E}_2) = 0$ @ $z=0$
where $\hat{n} = \hat{z}$

With the current sheet @ $z=0$, we can rationalize the fact that in region 1 ($z \geq 0$) there exists only an outgoing wave of the form $e^{-jk_0 z}$ and in region 2 ($z<0$) there also exists an outgoing wave of the form $e^{+jk_0 z}$. This rationalization is related to the <u>radiation condition</u> or alternatively to <u>causality</u>.

The total fields in each region are then –

$$\bar{E}_2 = \hat{x}\, D e^{jk_0 z} + \hat{y}\, B e^{jk_0 z} \qquad z \le 0$$

$$\bar{E}_1 = \hat{x}\, C e^{-jk_0 z} + \hat{y}\, A e^{-jk_0 z} \qquad z \ge 0$$

Applying B.C. →

$$\hat{z} \times \left[(\hat{x}\, C + \hat{y}\, A) - (\hat{x}\, D + \hat{y}\, B) \right] = 0$$

or $\underline{\underline{A = B}}$ & $\underline{\underline{C = D}}$ (E_{tan} is continuous)

The other B.C. is $\hat{n} \times (\bar{H}_1 - \bar{H}_2) = \bar{J}_s$

The $\bar{H}$ field in each region may be obtained from $\bar{E}$ and Maxwell's equations since

$$\nabla \times \bar{E} = -j\omega\mu_0 \bar{H} \quad \text{where} \quad \nabla \times \bar{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix}$$

$$= -\hat{x}\left(\frac{\partial E_y}{\partial z}\right) - \hat{y}\left(-\frac{\partial E_x}{\partial z}\right)$$

∴

$$-j\omega\mu_0 \bar{H} = -\hat{x}\frac{\partial E_y}{\partial z} + \hat{y}\frac{\partial E_x}{\partial z}$$

or

$$\bar{H} = \hat{x}\frac{1}{j\omega\mu_0}\frac{\partial E_y}{\partial z} - \hat{y}\frac{1}{j\omega\mu_0}\frac{\partial E_x}{\partial z}$$

in region 2 $\rightarrow$ $\bar{H}_2 = \hat{x}\, A \frac{jk_o}{j\omega\mu_o} e^{jk_o z} - \hat{y} \frac{jk_o}{j\omega\mu_o} C e^{jk_o z}$

in region 1 $\rightarrow$ $\bar{H}_1 = \hat{x}\, A \frac{-jk_o}{j\omega\mu_o} e^{-jk_o z} - \hat{y} \frac{-jk_o}{j\omega\mu_o} C e^{-jk_o z}$

@ $z=0$, $\hat{z} \times (\bar{H}_1 - \bar{H}_2) = -\hat{x} J_{s_o}$

Giving -

$$-\hat{y}\, A \frac{k_o}{\omega\mu_o} - \hat{x} \frac{k_o}{\omega\mu_o} C - \left[\hat{y} A \frac{k_o}{\omega\mu_o} + \hat{x} \frac{k_o}{\omega\mu_o} C\right] = -\hat{x} J_{s_o}$$

or/

$$\hat{x}: \quad -C - C = -\frac{\omega\mu_o}{k_o} J_{s_o} \Rightarrow \underline{\underline{C = +\frac{\omega\mu_o}{2k_o} J_{s_o}}}$$

$$\hat{y}: \quad -A \frac{k_o}{\omega\mu_o} - A \frac{k_o}{\omega\mu_o} = 0 \Rightarrow \underline{\underline{A = 0}}$$

Substituting back gives the solutions for the total fields in each region to be -

$$(z \le 0) \quad \underline{\underline{\bar{E}_2 = \hat{x} \frac{\omega\mu_o}{2k_o} J_{s_o} e^{+jk_o z}}} \qquad \underline{\underline{\bar{H}_2 = -\hat{y} \frac{J_{s_o}}{2} e^{+jk_o z}}}$$

$$(z \ge 0) \quad \underline{\underline{\bar{E}_1 = \hat{x} \frac{\omega\mu_o}{2k_o} J_{s_o} e^{-jk_o z}}} \qquad \underline{\underline{\bar{H}_1 = \hat{y} \frac{J_{s_o}}{2} e^{-jk_o z}}}$$

These results are the complete solutions to the fields which exist in space due to the infinite current sheet source.

We notice from the above solutions that $\bar{E}$ and $\bar{H}$ are perpendicular to one another. Also, in planes perpendicular to the direction of propagation ($\pm z$) the fields are constant, or "uniform," in value and direction. Therefore, these fields are called uniform plane waves.

Another characteristic property of this type of wave is the ratio of the orthogonal components of $\bar{E}$ & $\bar{H}$. Namely,

in region 1: $$\frac{E_{x1}}{H_{y1}} = \frac{\frac{\omega\mu_0}{2k_0} J_{s0}}{\frac{J_{s0}}{2}} = \eta_0$$

in region 2: $$\frac{E_{x2}}{H_{y2}} = -\eta_0$$

where $\eta_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}}$ and is the intrinsic impedance of the medium

Summarizing the characteristics of uniform plane waves –

- $\bar{E} \perp \bar{H}$
- $\frac{E_i}{H_j} = \pm \eta_o$
- the fields have constant direction, magnitude & phase in planes perpendicular to the direction of propagation (Both $\bar{E}$ & $\bar{H}$).