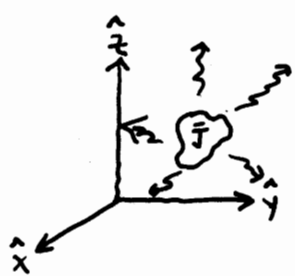


We will shortly require, in our numerical solution methods, the ability to find the fields produced by a given distribution of electric current density: source/field relationships.

Assuming the source is varying in a time-harmonic fashion with $e^{j\omega t}$, then Maxwell's equations may be written as



$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \quad (1)$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E} + \bar{J} \quad (2)$$

The objective is to solve for the fields produced by this current distribution \bar{J} . The solutions, as we shall see, may be expressed as a convolution integral.

Method 1: The Potential Method

Taking $\nabla \cdot (1) \Rightarrow \nabla \cdot \nabla \times \bar{E} = -j\omega\mu \nabla \cdot \bar{H}$ (homogeneous)

or $0 = \nabla \cdot \bar{H}$ ←

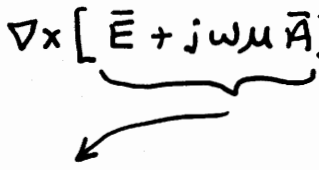
This type of field is called solenoidal.

A solenoidal field may be expressed as the curl of another vector (Helmholtz theorem)

giving
$$\vec{H} = \nabla \times \vec{A} \quad (3)$$

Sub. this result into (1) gives $\nabla \times \vec{E} = -j\omega\mu \nabla \times \vec{A}$

or
$$\nabla \times [\vec{E} + j\omega\mu \vec{A}] = 0$$



A vector field which is curl-free is called irrotational or conservative & can be expressed as the gradient of another scalar potential →

$$\vec{E} + j\omega\mu \vec{A} = -\nabla \Phi_e \quad (4)$$

↑
by convention

Now, sub. both (4) & (3) into (2) giving -

$$\nabla \times \nabla \times \vec{A} = j\omega\epsilon [-j\omega\mu \vec{A} - \nabla \Phi_e] + \vec{J}$$

vector i.d. $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

[where $\nabla^2 \vec{A}$ is defined on the cartesian components of \vec{A} only! ($\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z$)]

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} - \omega^2 \mu \epsilon \vec{A} + j\omega\epsilon \Phi_e = \vec{J}$$

rewriting - $\nabla^2 \vec{A} + k^2 \vec{A} - \nabla(\nabla \cdot \vec{A} + j\omega\epsilon \Phi_e) = -\vec{J}$

By the Helmholtz theorem, a vector field is uniquely specified (within an additive constant) when both the curl and divergence are specified.

- We've selected $\nabla \times \bar{A} = \bar{H}$
- Now choose $\nabla \cdot \bar{A} = -j\omega\epsilon\Phi_e$ to simplify the last eqn.

This choice is called the Lorentz Gauge.
For time-varying fields it is consistent w/ conservation of charge. [$\nabla \cdot \bar{A} = 0 \Rightarrow$ Coulomb Gauge.]

Substituting this last choice for $\nabla \cdot \bar{A}$ gives -

$$\underline{\underline{\nabla^2 \bar{A} + k^2 \bar{A} = -\bar{J}}} \quad (5)$$

A vector Helmholtz eqn. for \bar{A} . It is a set of 3 scalar equations - inhomogeneous O.D.E.'s

Once this equation has been solved for \bar{A} , then the fields are given as -

- $\bar{H} = \nabla \times \bar{A}$
- $\bar{E} = -j\omega\mu\bar{A} - \nabla\Phi_e$ or w/ $\Phi_e = -\frac{1}{j\omega\epsilon}\nabla \cdot \bar{A}$
- $\bar{E} = -j\omega\mu\bar{A} + \frac{1}{j\omega\epsilon}\nabla\nabla \cdot \bar{A}$

Method 2 : The Direct Method

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \quad (1) \quad ; \quad \nabla \times \bar{H} = j\omega\epsilon\bar{E} + \bar{J} \quad (2)$$

Take $\nabla \times (1) \rightarrow \nabla \times \nabla \times \bar{E} = -j\omega\mu \nabla \times \bar{H}$

Sub (2) into this giving - $\nabla \times \nabla \times \bar{E} = -j\omega\mu(j\omega\epsilon\bar{E} + \bar{J})$

or) $\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} - \omega^2\mu\epsilon\bar{E} = -j\omega\mu\bar{J} \quad (6)$

But what is $\nabla \cdot \bar{E}$? Take $(\nabla \cdot)$ of (2)

$$\nabla \cdot \nabla \times \bar{H} = 0 = j\omega\epsilon \nabla \cdot \bar{E} + \nabla \cdot \bar{J}$$

or) $\nabla \cdot \bar{E} = -\frac{1}{j\omega\epsilon} \nabla \cdot \bar{J}$

Sub. into (6) gives

$$-\frac{1}{j\omega\epsilon} \nabla \nabla \cdot \bar{J} - \nabla^2 \bar{E} - k^2 \bar{E} = -j\omega\mu \bar{J}$$

or) $\underline{\underline{\nabla^2 \bar{E} + k^2 \bar{E} = j\omega\mu \bar{J} - \frac{1}{j\omega\epsilon} \nabla \nabla \cdot \bar{J}}} \quad (7)$

Once \bar{E} has been solved, can find \bar{H} from Maxwell's equations.

In many other EM courses, primarily undergraduate, this is about the extent of the source/field concept.

Typically at this point, the current density \vec{J} would be assumed to have some functional form from which the fields could mathematically be computed using the Potential Method (5) & Maxwell's eqns.

However, in this course we will be solving for either ρ_e or \vec{J} ! They are the unknowns.

Once they have been found (either ρ_e or \vec{J} , depending on the problem), we can calculate all other fields, power, etc...

Problem - From (5) & (7) we don't have expressions of the form $\vec{A} = (\dots)$ or $\vec{E} = (\dots)$.

$$\text{Instead we have } \begin{aligned} (\nabla^2 + k^2) \vec{A} &= -\vec{J} \\ (\nabla^2 + k^2) \vec{E} &= -\frac{\nabla \nabla \cdot \vec{J} + k^2 \vec{J}}{j\omega\epsilon} \end{aligned}$$

Form of L.H.S is the same

Forcing fcts. are different.

⊗ To actually solve for \vec{A} or \vec{E} we must invert this linear operator $\nabla^2 + k^2$.