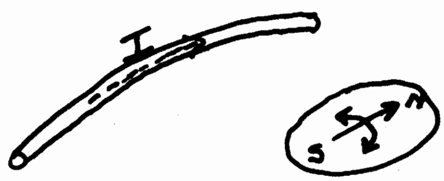
Maxwell's Equations & Boundary Conditions

The study of electromagnetics is based principally on two experimentally derived laws (axioms) and a conservation principle.

They are:
- Ampere's Law
- Faraday's Law
- conservation of charge

Ampere's Law : A current flowing in a wire produces a "magnetic field"



← compass needle moves

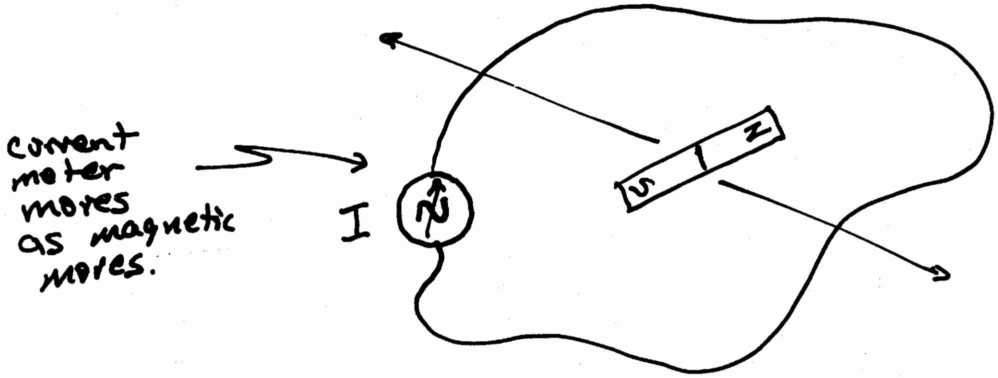
(*)! Maxwell's Addition: A time-varying electric field also produces a (time-varying) magnetic field. That is, its EM effects are like those of an electric current

Ampere's Law in integral form can be written as

$$\oint_{C(S)} \bar{H} \cdot d\bar{l} = \underbrace{\int_{S(C)} \bar{J} \cdot d\bar{s}}_{\text{electric current}} + \underbrace{\int_{S(C)} \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}}_{\text{displacement current}}$$

Both terms on RHS are "currents". Will see later that the addition of the displacement current term is a necessary consequence to satisfying conservation of charge!

Faraday's Law: is based on the experimentally observed fact that time-varying magnetic fields cause a current to flow in a wire. (Referred to as "induction" → current flow is "induced" in the wire.)



One can also keep the magnetic still & move the wire and still observe the same effect $\longrightarrow$ relativity.

This phenomenon was explained mathematically by stating that the

$$\text{EMF} \propto \frac{\partial \psi}{\partial t} \qquad \text{where } \psi = \text{magnetic "flux"}$$

or,

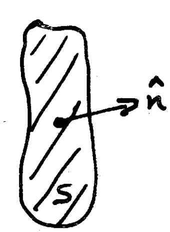
$$\oint_{C(S)} \bar{\mathcal{E}} \cdot d\bar{l} = -\int_{S(C)} \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

Conservation of charge:

By definition, the electric current in a filament of wire is given as

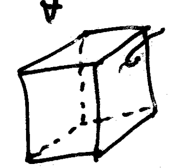
$$I = \frac{dQ}{dt} = \frac{d}{dt}(\text{charge})$$

Similarly for a surface S -



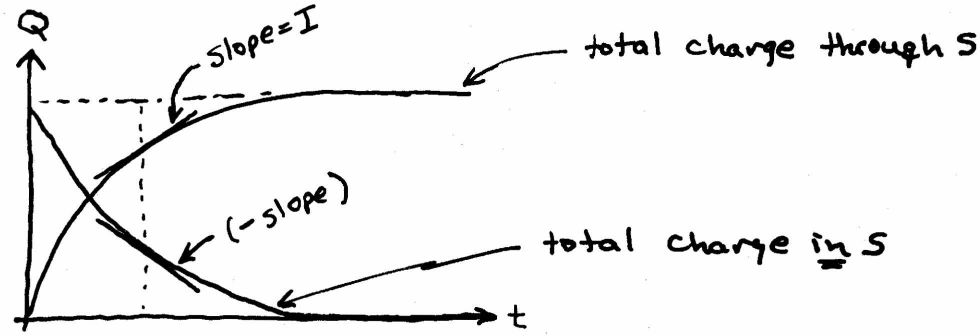
$$I = \frac{d}{dt}(\text{charge through S})$$

And for a closed surface, the current flowing out of the closed surface is



$$I = \frac{d}{dt}(\text{charge out through surface})$$

A fundamental postulate of physics is that charges can neither be created nor destroyed. That is, they are conserved.



Using this principle then, $I = -\frac{d}{dt} Q_{inside}$

The electric current can also be related to the current density $\bar{J}$ as

$$I = \int_S \bar{J} \cdot d\bar{s}$$ where S is some arbitrary surface.

or for a closed surface

$$I = \oint_S \bar{J} \cdot d\bar{s} .$$

Equating these expressions for the current gives

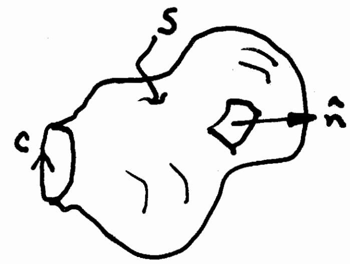
ρ_e = volume charge density

$$\oint_{S(V)} \bar{J} \cdot d\bar{s} = -\frac{d}{dt} Q_{inside} = -\int_{V(S)} \frac{\partial \rho_e}{\partial t} \, dv$$

continuity equation

The differential form of these basic equations can be derived using the divergence and Stokes' thms.

Faraday's Law → $\oint_{C(S)} \bar{E}\cdot d\bar{\ell} = -\int_{S(C)} \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$



by Stokes' thm $\oint_{C(S)} \bar{E}\cdot d\bar{\ell} = \int_{S(C)} \nabla\times\bar{E}\cdot d\bar{s}$

Substituting – $\int_{S(C)} \left[\nabla\times\bar{E} + \frac{\partial \bar{B}}{\partial t}\right]\cdot ds = 0$

Since this last expression is valid $\forall s \Rightarrow$

$$\nabla\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

Similarly for Ampere's Law →

$$\nabla\times\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

For the continuity equation –

$$\oint_{S(V)} \bar{J}\cdot d\bar{s} = -\int_{V(S)} \frac{\partial \rho_e}{\partial t}\, dv$$

Use the divergence thm. $\Rightarrow \oint_{S(V)} \bar{J} \cdot d\bar{s} = \int_{V(S)} \nabla \cdot \bar{J} \, dv$

Substituting: $\int_V \left[\nabla \cdot \bar{J} + \frac{\partial \rho_e}{\partial t} \right] dv = 0$

Since this last expression is valid $\forall\, V \Rightarrow$

$$\underline{\underline{\nabla \cdot \bar{J} = -\frac{\partial \rho_e}{\partial t}}}$$

The other two Maxwell's equations are Gauss' Laws $\rightarrow$

$$\oint_S \bar{D} \cdot d\bar{s} = Q \qquad \text{or/} \qquad \nabla \cdot \bar{D} = \rho_e$$

$$\oint_S \bar{B} \cdot d\bar{s} = 0 \qquad \text{or/} \qquad \nabla \cdot \bar{B} = 0$$

Phasors

For time harmonic fields in a linear medium, the time & spatial dependencies can be separated to give, for example,

$$\bar{\mathcal{E}}(\bar{r}, t) = Re\left\{ \bar{E}(\bar{r}) e^{j\omega t} \right\}$$

spatial dependence

Phasor

the maxwell's equations in time-harmonic form become

$$\nabla \times \bar{E} = -j\omega \bar{B} \qquad \nabla \cdot \bar{D} = \rho_e$$
$$\nabla \times \bar{H} = \bar{J} + j\omega \bar{D} \qquad \nabla \cdot \bar{B} = 0$$

and $$\nabla \cdot \bar{J} = -j\omega \rho_e$$

Typically in EM work, $\bar{J}$ & ρ_e are unknowns which need to be determined (rather than the fields), <u>or</u> they are assumed known (from which the fields can be calculated).

Assuming $\bar{J}$ & ρ_e are <u>known</u>, it appears we have 8 scalar equations in 12 unknowns. However, for $\omega \neq 0 \rightarrow$

$$\nabla \cdot \nabla \times \bar{E} = 0 = -j\omega \nabla \cdot \bar{B} \qquad \text{or} \qquad \underline{\nabla \cdot \bar{B} = 0}$$

and $$\nabla \cdot \nabla \times \bar{H} = 0 = \nabla \cdot \bar{J} + j\omega \nabla \cdot \bar{D}$$

but $$\nabla \cdot \bar{J} = -j\omega \rho_e \Rightarrow \underline{\nabla \cdot \bar{D} = \rho_e}$$

Gauss' Laws are <u>not</u> independent equations. Many criterion to be satisfied.

<u>Therefore</u>, we actually have <u>6</u> scalar equations in <u>12</u> unknowns. The additional 6 equations are supplied by the constitutive parameters. For an <u>isotropic</u> medium

$$\bar{D} = \epsilon \bar{E} \quad \& \quad \bar{B} = \mu \bar{H}$$

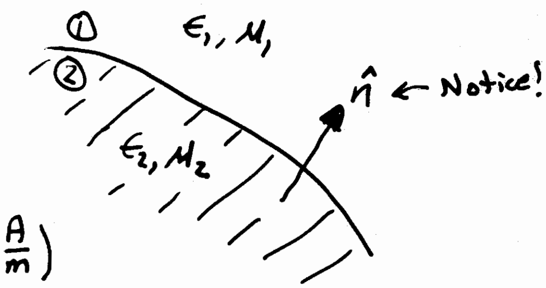
Boundary Conditions

For regions of space with differing constitutive parameters, we need boundary conditions in which to apply to the fields in order to solve Maxwell's p.d.e.'s

Where are these boundary conditions?
In Maxwell's equations themselves!

At the interface between 2 material media –

$$\hat{n} \times (\bar{E}_1 - \bar{E}_2) = 0$$
$$\hat{n} \times (\bar{H}_1 - \bar{H}_2) = \bar{J}_s$$



$\bar{J}_s$ = surface current density $\left(\frac{A}{m}\right)$

and,
$$\hat{n} \cdot (\bar{D}_1 - \bar{D}_2) = \rho_s$$
$$\hat{n} \cdot (\bar{B}_1 - \bar{B}_2) = 0$$

ρ_s = surface charge density $\left(\frac{C}{m^2}\right)$

When region 2 is a perfect electrical conductor (pec) meaning $\sigma \to \infty$ for conduction current densities defined by $\bar{J} = \sigma \bar{E}$, then

$$\hat{n} \times \bar{E}_1 = 0 \qquad \hat{n} \cdot \bar{D}_1 = \rho_s$$
$$\hat{n} \times \bar{H}_1 = \bar{J}_s \qquad \hat{n} \cdot \bar{B}_1 = 0$$