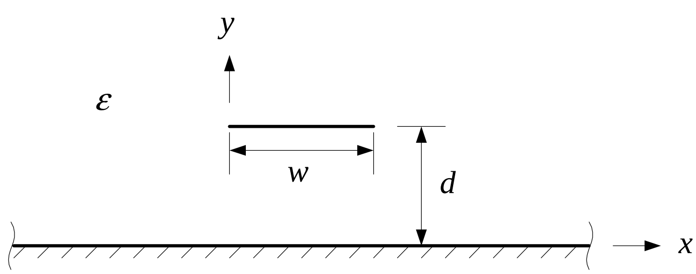
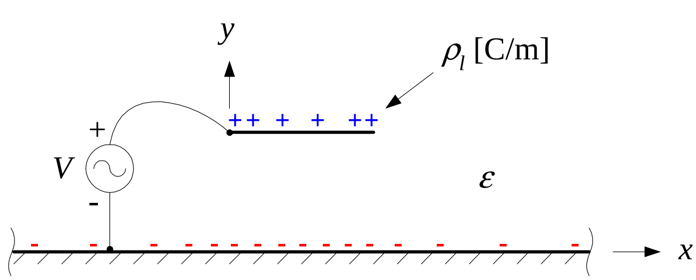
Lecture xx: MM Solution for a Microstrip Using Pulse Expansion-Point Match

The second example we will consider is the quasi-static moment method solution of a microstrip. For simplicity, we will assume that the entire upper half space is filled with the same material.



We'll imagine that a time-harmonic voltage source has been connected between the strip and the ground plane. This will cause a non-uniform charge distribution on the strip as well as the ground plane.

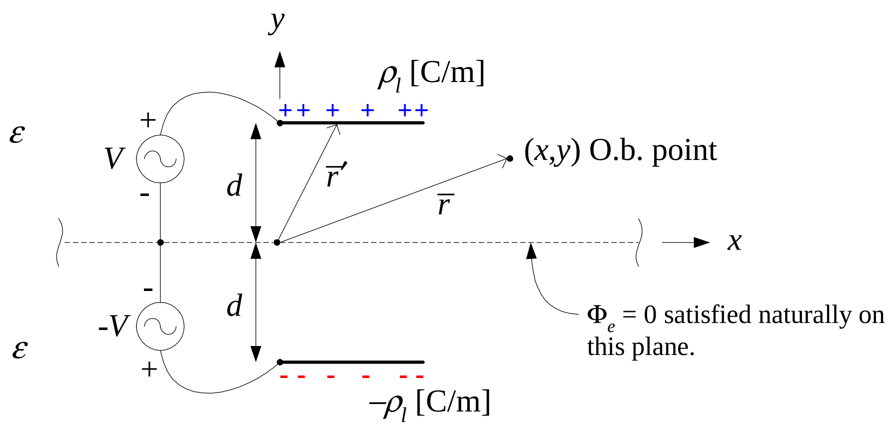


Provided the height and width of the microstrip are electrically very small, we can expect that a quasi-static solution will be quite accurate.

Quasi-static means that the time harmonic fields are simply those of the static solution, but are oscillating sinusoidally in time. In other words, there are no radiation fields present so all electric field lines begin and end on charges.

Integral Equation

Next, we'll employ the image theory method to create an equivalent problem for the upper half space ($y \geq 0$).



You've learned that the potential at a point $\bar{r}$ in a homogeneous space produced by a line charge density $\rho_l(\bar{r}')$ is given by

$$\begin{aligned}\Phi_e(\bar{r}) &= \frac{1}{2\pi\varepsilon}\int_{C'}\rho_l(\bar{r}')\ln\left(\frac{1}{|\bar{r}-\bar{r}'|}\right)dl' \\ &= -\frac{1}{2\pi\varepsilon}\int_{C'}\rho_l(\bar{r}')\ln\left(\sqrt{(x-x')^2+(y-y')^2}\right)dl'\end{aligned} \tag{1}$$

Note that we must integrate along all contours C' that have surface charge. Hence C' includes both the strip and its image.

Now, we will apply the boundary condition that on the top strip

$$\Phi_e(\bar{r}) = V \qquad \forall \bar{r} \in \{\text{upper strip}\} \tag{2}$$

Using (2) in (1) and accounting for both the $+\rho_l$ and $-\rho_l$ charge distributions on the strips then

$$V = -\frac{1}{2\pi\varepsilon}\left\{\int_{\text{top}} \rho_l(\bar{r}')\ln\left(\sqrt{(x-x')^2+(d-d)^2}\right)dl' + \int_{\text{bottom}} -\rho_l(\bar{r}')\ln\left(\sqrt{(x-x')^2+(d+d)^2}\right)dl'\right\}$$

or

$$\boxed{V = -\frac{1}{2\pi\varepsilon}\int_0^w \rho_l(\bar{r}')\left[\ln\left(|x-x'|\right) - \ln\left(\sqrt{(x-x')^2+4d^2}\right)\right]dx'} \tag{3}$$

This is the integral equation for the line charge density on the strip.

(Another approach to deriving this integral equation for the charge density would be to first find the Green's function and then convolve it with the forcing function.)

Pulse Expansion-Point Match

For a simple MM solution, we will use a pulse expansion for the charge density

$$\rho_l(\bar{r}') \approx \sum_{n=1}^{N} \alpha_n P_n(x'; x_{n-1}, x_n) \tag{4}$$

and point match the discretized integral equation.

Substituting (4) into (3) gives

$$V = -\frac{1}{2\pi\varepsilon}\int_0^w \left[\sum_{n=1}^{N} \alpha_n P_n\left(x'; x_{n-1}, x_n\right)\right] G\left(\left|x - x'\right|\right) dx' \qquad (5)$$

where $$\boxed{G\left(\left|x-x'\right|\right) = \ln\left(\left|x-x'\right|\right) - \ln\left(\sqrt{\left(x-x'\right)^2 + 4d^2}\right)} \qquad (6)$$

In (5), we can interchange the order of integration and summation, since these are linear operators, except perhaps when $x = x'$. Assuming this isn't the case, then (5) becomes

$$V = -\frac{1}{2\pi\varepsilon}\sum_{n=1}^{N}\alpha_n \int_{x_{n-1}}^{x_n} G\left(\left|x-x'\right|\right) dx' \qquad (7)$$

We'll generate N linearly independent, constant coefficient equations by point matching (7) at the centroid of every segment in the discretized strip model. We'll denote these points as $\bar{r}_m = \hat{x}x_m + \hat{y}d$.

Point matching (7) at these points $\bar{r}_m$ gives

$$\boxed{V = -\frac{1}{2\pi\varepsilon}\sum_{n=1}^{N}\alpha_n \int_{x_{n-1}}^{x_n} G\left(\left|x_m - x'\right|\right) dx' \qquad m = 1,\ldots,N} \qquad (8)$$

which is a matrix equation of the form

$$\underbrace{\left[V_m\right]}_{N\times 1} = \underbrace{\left[Z_{mn}\right]}_{N\times N}\cdot\underbrace{\left[\alpha_n\right]}_{N\times 1} \qquad (9)$$

where $$V_m = V \qquad (10a)$$

$$\alpha_n = \alpha_n \qquad (10b)$$

and $$Z_{mn} = -\frac{1}{2\pi\varepsilon}\int_{x_{n-1}}^{x_n} G\left(\left|x_m - x'\right|\right) dx' \qquad m \neq n \tag{10c}$$

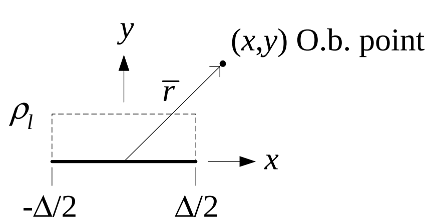
As before, the process for obtaining a numerical solution from (9) is to "fill" $[V]$ and $[Z]$, then solve this system of equations for the line charge density coefficients $[\alpha]$. In particular,

- For $[V]$ – choose V = 1 V, for example.
- For $[Z]$ – compute (10c) analytically, if possible, or use numerical integration.

In this particular problem, we are able to evaluate (10c) analytically since a simple anti-derivative of the integrand is available.

Analytical Evaluation of Z_{mn}

To accomplish this analytical evaluation of (10c), we will begin with a segment of width Δ located at the origin that is supporting a uniform line charge density ρ_l:



The electrostatic potential at point $\bar{r}$ produced by this "pulse" of line charge density is

$$\Phi_e(\bar{r}) = -\frac{\rho_l}{2\pi\varepsilon}\int_{-\Delta/2}^{\Delta/2} \ln\left[\sqrt{(x-x')^2+y^2}\right]dx' \tag{11}$$

When $\bar{r}$ does not lie anywhere on this strip, the potential is

$$\begin{aligned}\Phi_e(\bar{r}) = -\frac{\rho_l}{2\pi\varepsilon}\Big\{&(x+\Delta/2)\ln\left[(x+\Delta/2)^2+y^2\right]\\ &-(x-\Delta/2)\ln\left[(x-\Delta/2)^2+y^2\right]-2\Delta\\ &+2y\left[\tan^{-1}\left(\frac{x+\Delta/2}{y}\right)-\tan^{-1}\left(\frac{x-\Delta/2}{y}\right)\right]\Big\}\end{aligned} \tag{12}$$

Using (12) in (10c), it can be shown that for $m \neq n$

$$\boxed{\begin{aligned}Z_{mn} = -\frac{1}{4\pi\varepsilon}\Bigg\{&(\Delta_{mn}+\Delta/2)\ln\left[\frac{(\Delta_{mn}+\Delta/2)^2}{(\Delta_{mn}+\Delta/2)^2+4d^2}\right]\\ &-(\Delta_{mn}-\Delta/2)\ln\left[\frac{(\Delta_{mn}-\Delta/2)^2}{(\Delta_{mn}-\Delta/2)^2+4d^2}\right]\\ &-4d\left[\tan^{-1}\left(\frac{\Delta_{mn}+\Delta/2}{2d}\right)-\tan^{-1}\left(\frac{\Delta_{mn}-\Delta/2}{2d}\right)\right]\Bigg\}\end{aligned}} \tag{13}$$

where

$$\Delta_{mn} \equiv x_m - x_n \tag{14}$$

Evaluation of Z_{mm}

A more nettlesome situation occurs if the observation point lies on the segment. This will happen every time $m = n$ while the matrix $[Z]$ is being filled.

For these "self-cell" evaluations, the observation point will be located at the center of the segment in the chosen point-matching scheme

Referring to the figure above, if $x = y = 0$ (at the center of the strip) it can be shown that

$$\Phi_e(0) = \frac{\rho_l}{2\pi\varepsilon}\left[1 - \ln(\Delta/2)\right] \tag{15}$$

Consequently, from this result it can be shown that for $m = n$

$$\boxed{\begin{aligned} Z_{mm} &= \frac{\Delta}{2\pi\varepsilon}\left[1 - \ln(\Delta/2)\right] \\ &\quad + \frac{1}{4\pi\varepsilon}\left\{\Delta \ln\left[(\Delta/2)^2 + 4d^2\right] - 2\Delta + 8d \tan^{-1}\left(\frac{\Delta}{4d}\right)\right\} \end{aligned}} \tag{16}$$