Lecture 5 – Plane Waves in Lossless Media at Oblique Angles

In the previous lecture, we examined plane wave solutions to the wave equation. These solutions considered the rather simplistic case where $\bar{E}$ polarized in $\hat{a}_x$ direction only, with wave propagation only in $\pm\hat{a}_z$ directions.

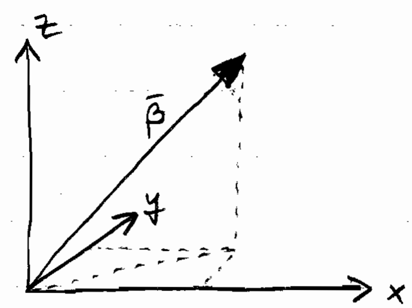
$$\bar{E}(z) = \hat{a}_x\left(E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z}\right) \equiv \hat{a}_x\left[E_x^+(z) + E_x^-(z)\right] \quad (1)$$

The corresponding $\bar{H}$ was found from Faraday's law to be

$$\bar{H}(z) = \hat{a}_y \frac{1}{\eta}\left(E_0^+ e^{-j\beta z} - E_0^- e^{+j\beta z}\right) \equiv \hat{a}_y\left[H_y^+(z) - H_y^-(z)\right] \quad (2)$$

These solutions were called plane waves, in planes ⊥ to the direction of propagation ($\pm z$), there is no variation in $\bar{E}$ or $\bar{H}$.

We will now consider the more general case where the plane wave is propagating in an arbitrary direction, $\hat{\beta}$.



The vector $\bar{\beta}$ is called the wave vector and in the Cartesian coordinate system

$$\bar{\beta} = \hat{\beta}\beta = \hat{a}_x\beta_x + \hat{a}_y\beta_y + \hat{a}_z\beta_z \quad (3)$$

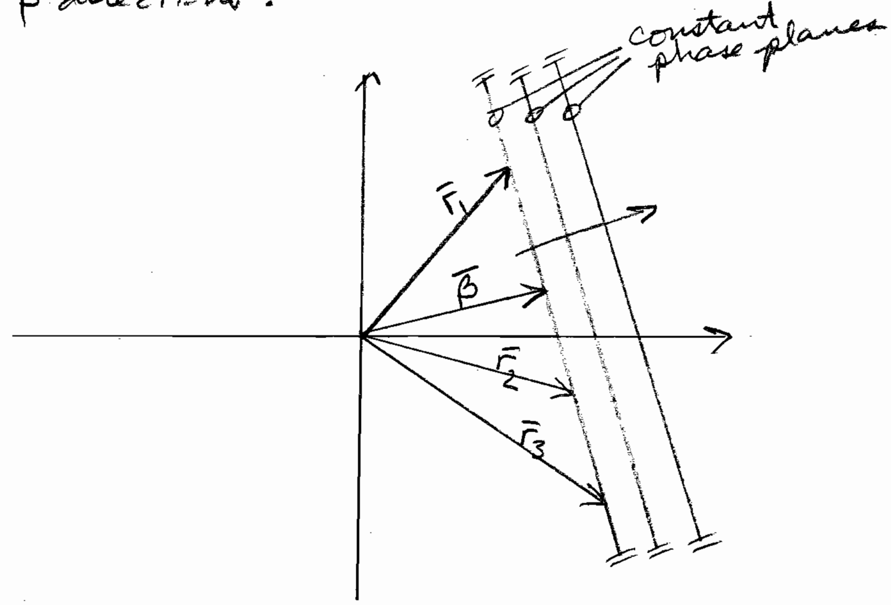
For a plane wave propagation, the equi-phase surfaces of this wave must be planes ⊥ to the direction of propagation. Mathematically, this can be written as

$$\bar{E}(\bar{r}) = \hat{e}\, E_o\, e^{\mp j \bar{\beta} \cdot \bar{r}} \tag{4}$$

$$\bar{H}(\bar{r}) = \hat{h}\, H_o\, e^{\mp j \bar{\beta} \cdot \bar{r}} \tag{5}$$

for propagation in the $+\hat{\beta}$ & $-\hat{\beta}$ directions. The $\hat{e}$ & $\hat{h}$ unit vectors point in the direction of $\bar{E}$ & $\bar{H}$, respectively. For a plane wave, $\bar{E} \perp \bar{H}$, & both must be ⊥ to direction of propagation.

The phase terms $e^{\mp j\bar{\beta}\cdot\bar{r}}$ indicate plane waves propagating in the $\pm\hat{\beta}$ directions:



The projection of $\bar{r}_1$ onto β, $\bar{r}_2$ onto $\bar{\beta}$, and $\bar{r}_3$ onto $\bar{\beta}$ are all equal because this is a plane wave:

$$\bar{\beta}\cdot\bar{r}_1 = \bar{\beta}\cdot\bar{r}_2 = \bar{\beta}\cdot\bar{r}_3 = C \text{ (complex constant)} \qquad (6)$$

Maxwell's Equations for Plane Waves

For functions of the form $e^{\pm\bar{\gamma}\cdot\bar{r}}$, such as plane waves, it can be shown that

- $\nabla e^{\pm\bar{\gamma}\cdot\bar{r}} = \pm\bar{\gamma}\, e^{\pm\bar{\gamma}\cdot\bar{r}}$
- $\nabla\cdot\hat{a}\, e^{\pm\bar{\gamma}\cdot\bar{r}} = \pm\bar{\gamma}\cdot\hat{a}\, e^{\pm\bar{\gamma}\cdot\bar{r}}$
- $\nabla\times\hat{a}\, e^{\pm\bar{\gamma}\cdot\bar{r}}$

From these results, we can deduce that as far as plane wave solutions to Maxwell's equations are concerned, we can replace:

$$\nabla \rightarrow \pm\bar{\gamma} \qquad (7)$$

for plane waves w/ dependence $e^{\pm\bar{\gamma}\cdot\bar{r}}$.

For example, Let's consider plane wave propagation in a lossless, simple, sourcefree space with propagation $e^{\mp j\bar{\beta}\cdot\bar{r}}$. It can be shown that Maxwells equations can be written as:

- Faraday's law: $\nabla\times\bar{E} = -j\omega\mu\bar{H}$

w/ $\nabla \rightarrow \mp j\bar{\beta}$ then

$$\mp j\bar{\beta} \times \bar{E} = -j\omega\mu\bar{H}$$

or $$\underline{\bar{\beta} \times \bar{E} = \pm\,\omega\mu\bar{H}} \tag{8}$$

- Ampère's law: $\nabla \times \bar{H} = j\omega\epsilon\bar{E}$

w/ $\nabla \rightarrow \mp j\bar{\beta}$ then $\mp j\bar{\beta} \times \bar{H} = j\omega\epsilon\bar{E}$

or $$\underline{\bar{\beta} \times \bar{H} = \mp\,\omega\epsilon\bar{E}} \tag{9}$$

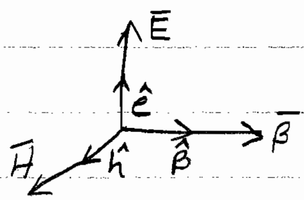
- Gauss's laws: $\nabla \cdot \bar{D} = 0$ or $\nabla \cdot \bar{E} = 0$

w/ $\nabla \rightarrow \mp j\bar{\beta}$ then $\underline{\bar{\beta} \cdot \bar{E} = 0}$ (10)

Similary, for $\nabla \cdot \bar{B} = 0 \Rightarrow \underline{\bar{\beta} \cdot \bar{H} = 0}$ (11)

These four equations (8)-(11) are very important in electromagnetics. We will use them extensively. These equations also provide valuable information for the general behavior of plane waves in a lossless, simple material.

Eqns. (8) & (9) tell us that $\bar{E} \perp \bar{H}$, while (10) & (11) tell us that E & $\bar{H}$ are both $\perp$ to the direction of propagation.



Dispersion Relationship

Taking $(\bar{\beta}\times)$ (8) we find

$$\bar{\beta}\times(\bar{\beta}\times\bar{E}) = \pm\omega\mu\bar{\beta}\times\bar{H} \tag{12}$$

substituting (9) in the RHS

$$\bar{\beta}\times(\bar{\beta}\times\bar{E}) = \pm\omega\mu(\mp\omega\epsilon\bar{E}) = -\omega^2\mu\epsilon\bar{E}$$

or

$$\underset{\bar{a}}{\bar{\beta}}\times(\underset{\bar{b}}{\bar{\beta}}\times\underset{\bar{c}}{\bar{E}}) + \beta^2\bar{E} = 0 \tag{13}$$

We'll apply the vector id. $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$ to the LHS of (13) giving

$$(\bar{\beta}\cdot\bar{E})\bar{\beta} - (\bar{\beta}\cdot\bar{\beta})\bar{E} + \beta^2\bar{E} = 0$$

From (10), $\bar{\beta}\cdot\bar{E} = 0$ so that

$$(\bar{\beta}\cdot\bar{\beta} - \beta^2)\bar{E} = 0. \tag{14}$$

A non-trivial solution requires that

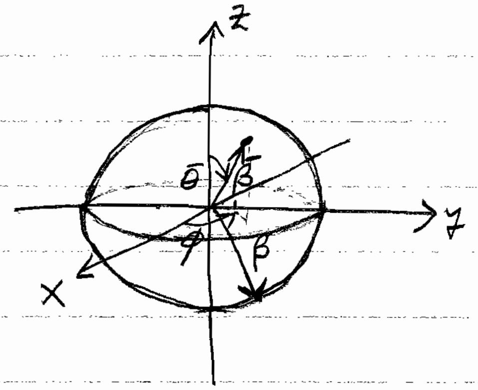
$$\underline{\bar{\beta}\cdot\bar{\beta} = \beta^2} \tag{15}$$

This is the dispersion relation for plane waves propagating in simple material. In cartesian coords. where

$$\bar{\beta} = \bar{a}_x\beta_x + \bar{a}_y\beta_y + \bar{a}_z\beta_z$$

then, $$\underline{\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2} \tag{16}$$

This is an equation for a sphere. (15) states that $\bar{\beta}$ lies on the surface of a sphere.



As the direction of propagation changes, the components of β change, but the tip of $\bar{\beta}$ remains on the sphere. A material in which the dispersion surface is a sphere is an isotropic space. From (15) or (16) we have one scalar equation that allows us to find one of the three comp's of $\bar{\beta}$. The other two require more information from the problem such as source conditions or boundary conditions.

In the case of the spherical coordinates θ & ϕ shown above, referring to eqn. (II-13b)

$$\beta_x = \beta \sin\theta \cos\phi$$
$$\beta_y = \beta \sin\theta \sin\phi$$
and $$\beta_z = \beta \cos\theta$$