In the previous lecture, we examined plane wave solutions to the wave equation. These solutions considered the rather simplistic case where $\mathbf{E}$ polarized in $\hat{x}$ direction only, with wave propagation only in $\pm \hat{z}$ directions.

$$\mathbf{E}(z) = \hat{x} \left( E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z} \right) \equiv \hat{x} \left[ E_x^+(z) + E_x^-(z) \right] \quad (1)$$

The corresponding $\mathbf{H}$ was found from Faraday's law to be

$$\mathbf{H}(z) = \hat{y} \frac{1}{\eta} \left( E_0^+ e^{-j\beta z} - E_0^- e^{+j\beta z} \right) \equiv \hat{y} \left[ H_y^+(z) - H_y^-(z) \right] \quad (2)$$

These solutions were called plane waves, the planes $\perp$ to the direction of propagation ($\pm z$), there is no variation in $\mathbf{E}$ or $\mathbf{H}$.

We will now consider the more general case, where the plane wave is propagating in an arbitrary direction, $\beta$.

\[
\begin{align*}
\mathbf{\beta} &= \hat{\beta} \beta = \hat{x} \beta_x + \hat{y} \beta_y + \hat{z} \beta_z
\end{align*}
\]
For a plane wave propagation, the equi-phase surfaces of this wave must be planes \( \pm \hat{\beta} \) to the direction of propagation. Mathematically, this can be written as

\[
E(r) = \hat{e} E_0 e^{\mp j \beta \cdot \hat{r}}
\]

\[
H(r) = \hat{n} H_0 e^{\mp j \beta \cdot \hat{r}}
\]

For propagation in the \( \hat{\beta} \) \( \pm \beta \) directions. The \( \hat{e} \) and \( \hat{n} \) are unit vectors pointing in the direction of \( E \) and \( H \), respectively. For a plane wave, \( E \pm H \) both must be \( \pm \hat{\beta} \) to the direction of propagation.

The phase terms \( e^{\mp j \beta \cdot \hat{r}} \) indicate plane waves propagating in the \( \pm \hat{\beta} \) directions.

The projection of \( E_1 \) onto \( \hat{\beta} \), \( E_2 \) onto \( \hat{\beta} \), and \( E_3 \) onto \( \hat{\beta} \) are all equal because this is a plane wave.
Maxwell's Equations for Plane Waves

For functions of the form $e^{\pm \varphi \cdot \mathbf{r}}$, such as plane waves, it can be shown that:

- $\nabla \cdot \mathbf{E}^{\pm \varphi \cdot \mathbf{r}} = \pm \varphi \mathbf{E}^{\pm \varphi \cdot \mathbf{r}}$
- $\nabla \cdot \mathbf{B}^{\pm \varphi \cdot \mathbf{r}} = \pm \varphi \mathbf{B}^{\pm \varphi \cdot \mathbf{r}}$
- $\nabla \times \mathbf{E}^{\pm \varphi \cdot \mathbf{r}} = \pm \varphi \mathbf{B}^{\pm \varphi \cdot \mathbf{r}}$
- $\nabla \times \mathbf{B}^{\pm \varphi \cdot \mathbf{r}} = \pm \varphi \mathbf{E}^{\pm \varphi \cdot \mathbf{r}}$

From these results, we can deduce that as far as plane wave solutions to Maxwell's equations are concerned, we can replace:

$$\nabla \rightarrow \pm \varphi$$  \hspace{1cm} (7)

for plane waves with dependence $e^{\pm \varphi \cdot \mathbf{r}}$.

For example, let's consider plane wave propagation in a lossless, simple, source-free space with propagation $e^{\pm j \beta \cdot \mathbf{r}}$. It can be shown that Maxwell's equations can be written as:

- Faraday's law: $\nabla \times \mathbf{E} = -j \omega \mu \mathbf{H}$

with $\nabla \rightarrow \pm j \beta$, then
\[ \vec{J} = \vec{J}_0 + \vec{J}_1 = -\gamma \omega \mu \vec{H} \]

or
\[ \vec{B} \times \vec{E} = \pm \omega \mu \vec{H} \]  

(8)

- **Ampère's law:** \[ \nabla \times \vec{H} = j \omega \varepsilon \vec{E} \]

\[ \omega \nabla \rightarrow \mp j \vec{B} \] then \[ \mp j \vec{B} \times \vec{H} = \frac{1}{2} \omega \varepsilon \vec{E} \]

or
\[ \vec{B} \times \vec{H} = \mp \omega \varepsilon \vec{E} \]  

(9)

- **Gauss's laws:** \[ \nabla \cdot \vec{D} = 0 \] or \[ \nabla \cdot \vec{E} = 0 \]

\[ \omega \nabla \rightarrow \mp j \vec{B} \] then \[ \vec{B} \cdot \vec{E} = 0 \]  

(10)

Similarly, for \[ \nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} \cdot \vec{H} = 0 \]  

(11)

These four equations (8) - (11) are very important in electromagnetics. We will use them extensively.

These equations also provide valuable information for the general behavior of plane waves in a lossless, simple material.

Eqs. (8) and (9) tell us that \( \vec{E} \perp \vec{H} \), while (10) and (11) tell us that \( \vec{E} \parallel \vec{H} \) are both \( \perp \) to the direction of propagation.
Dispersion Relationship

Taking (\( \vec{\beta} \times \vec{E} \)) (8) we find

\[
\vec{\beta} \times (\vec{\beta} \times \vec{E}) = \pm \omega \mu (\vec{\beta} \times \vec{H})
\]

Substituting (9) in the RHS

\[
\vec{\beta} \times (\vec{\beta} \times \vec{E}) = \pm \omega \mu (\vec{\beta} \times \vec{E}) = -\omega^2 \mu \vec{E}
\]

or

\[
\frac{\vec{\beta} \times (\vec{\beta} \times \vec{E}) + \beta^2 \vec{E}}{\alpha} = 0
\]

(13)

We'll apply the vector id., \( \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} \)
to the RHS of (13) giving

\[
(\vec{\beta} \cdot \vec{E}) \vec{\beta} - (\vec{\beta} \cdot \vec{\beta}) \vec{E} + \beta^2 \vec{E} = 0
\]

From (10), \( \vec{\beta} \cdot \vec{E} = 0 \) so that

\[
(\vec{\beta} \cdot \vec{\beta} - \beta^2) \vec{E} = 0.
\]

(14)

A non-trivial solution requires that

\[
\vec{\beta} \cdot \vec{\beta} = \beta^2
\]

(15)

This is the dispersion relation for plane waves propagating in simple materials. In cartesian coords, where

\[
\vec{\beta} = \vec{a}_x \beta_x + \vec{a}_y \beta_y + \vec{a}_z \beta_z
\]
Then, \[ \beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 \] \hspace{1cm} (14)

This is an equation for a sphere. (15) states that \( \beta \) lies on the surface of a sphere.

As the direction of propagation changes, the components of \( \beta \) change, but the tip of \( \beta \) remains on the sphere.

A material in which the dispersion surface is a sphere is an anisotropic space. From (15) or (16) we have one scalar equation that allows us to find one of the three cosines of \( \beta \). The other two require more information from the problem such as source conditions or boundary conditions.

In the case of the spherical coordinates \( \Theta \) and \( \phi \) shown above, referring to eqn. (II-13b)

\[
\begin{align*}
\beta_x &= \beta \sin \Theta \cos \phi \\
\beta_y &= \beta \sin \Theta \sin \phi \\
\text{and} \quad \beta_z &= \beta \cos \Theta
\end{align*}
\]