

In the previous lecture, we examined plane wave solutions to the wave equation. These solutions considered the rather simplistic case where \vec{E} polarized in \hat{a}_x direction only, with wave propagation only in $\pm \hat{a}_z$ directions.

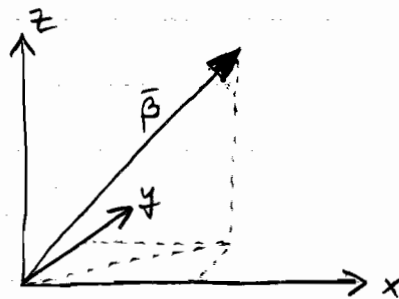
$$\vec{E}(z) = \hat{a}_x \left(E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z} \right) \equiv \hat{a}_x \left[E_x^+(z) + E_x^-(z) \right] \quad (1)$$

The corresponding \vec{H} was found from Faraday's law to be

$$\vec{H}(z) = \hat{a}_y \frac{1}{\eta} \left(E_0^+ e^{-j\beta z} - E_0^- e^{+j\beta z} \right) \equiv \hat{a}_y \left[H_y^+(z) - H_y^-(z) \right] \quad (2)$$

These solutions were called plane waves, in planes \perp to the direction of propagation ($\pm z$), there is no variation in \vec{E} or \vec{H} .

We will now consider the more general case where the plane wave is propagating in an arbitrary direction, $\hat{\beta}$.



The vector $\vec{\beta}$ is called the wave vector and in ^{the} Cartesian coordinate system

$$\vec{\beta} = \hat{\beta} \beta = \hat{a}_x \beta_x + \hat{a}_y \beta_y + \hat{a}_z \beta_z \quad (3)$$

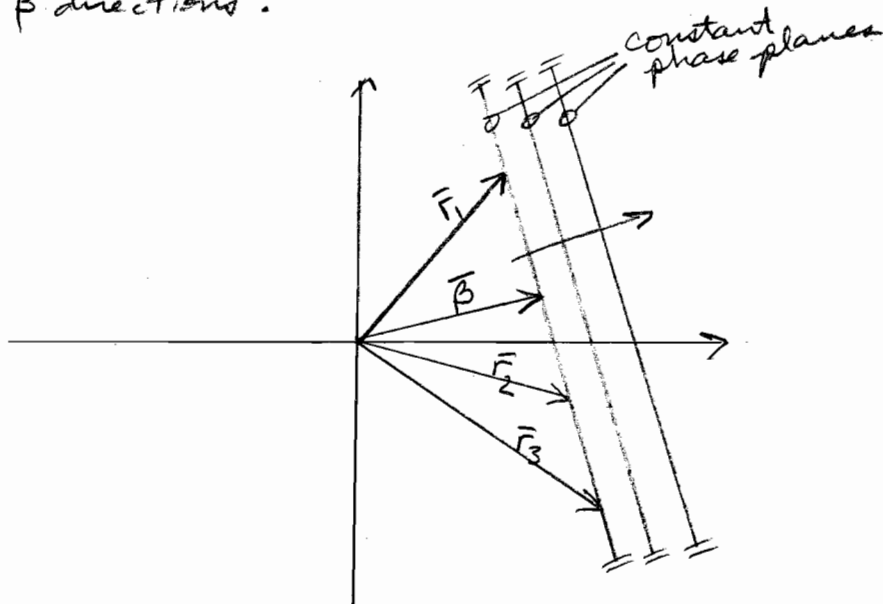
For a plane wave propagation, the equi-phase surfaces of this wave must be planes \perp to the direction of propagation. Mathematically, this can be written as

$$\vec{E}(\vec{r}) = \hat{e} E_0 e^{\mp j\vec{\beta} \cdot \vec{r}} \quad (4)$$

$$\vec{H}(\vec{r}) = \hat{h} H_0 e^{\mp j\vec{\beta} \cdot \vec{r}} \quad (5)$$

For propagation in the $+\hat{\beta} : -\hat{\beta}$ directions. The $\hat{e} : \hat{h}$ unit vectors point in the direction of $\vec{E} : \vec{H}$, respectively. For a plane wave, $\vec{E} \perp \vec{H}$, both must be \perp to direction of propagation.

The phase terms $e^{\mp j\vec{\beta} \cdot \vec{r}}$ indicate plane waves propagating in the $\pm \hat{\beta}$ directions:



The projection of \vec{r}_1 onto $\vec{\beta}$, \vec{r}_2 onto $\vec{\beta}$, and \vec{r}_3 onto $\vec{\beta}$ are all equal because this is a plane wave:

$$\vec{\beta} \cdot \vec{r}_1 = \vec{\beta} \cdot \vec{r}_2 = \vec{\beta} \cdot \vec{r}_3 = C \text{ (complex constant)} \quad (6)$$

Maxwell's Equations for Plane Waves

For functions of the form $e^{\pm \vec{\gamma} \cdot \vec{r}}$, such as plane waves, it can be shown that

- $\nabla e^{\pm \vec{\gamma} \cdot \vec{r}} = \pm \vec{\gamma} e^{\pm \vec{\gamma} \cdot \vec{r}}$
- $\nabla \cdot \hat{a} e^{\pm \vec{\gamma} \cdot \vec{r}} = \pm \vec{\gamma} \cdot \hat{a} e^{\pm \vec{\gamma} \cdot \vec{r}}$
- $\nabla \times \hat{a} e^{\pm \vec{\gamma} \cdot \vec{r}}$

From these results, we can deduce that as far as plane wave solutions to Maxwell's equations are concerned, we can replace:

$$\nabla \rightarrow \pm \vec{\gamma} \quad (7)$$

for plane waves w/ dependence $e^{\pm \vec{\gamma} \cdot \vec{r}}$.

For example, let's consider plane wave propagation in a lossless, simple, source-free space with propagation $e^{\mp j \vec{\beta} \cdot \vec{r}}$. It can be shown that Maxwell's equations can be written as:

- Faraday's law: $\nabla \times \vec{E} = -j\omega \mu \vec{H}$

w/ $\nabla \rightarrow \mp j \vec{\beta}$ then

$$\nabla \times \vec{B} = -j\omega\mu\vec{H}$$

or
$$\underline{\vec{B} \times \vec{E} = \pm \omega\mu\vec{H}} \quad (8)$$

• Ampere's law: $\nabla \times \vec{H} = j\omega\epsilon\vec{E}$

w/ $\nabla \rightarrow \mp j\vec{\beta}$ then $\nabla \times \vec{H} = j\omega\epsilon\vec{E}$

or
$$\underline{\vec{\beta} \times \vec{H} = \mp \omega\epsilon\vec{E}} \quad (9)$$

• Gauss's laws: $\nabla \cdot \vec{D} = 0$ or $\nabla \cdot \vec{E} = 0$

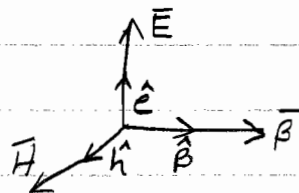
w/ $\nabla \rightarrow \mp j\vec{\beta}$ then $\underline{\vec{\beta} \cdot \vec{E} = 0} \quad (10)$

Similarly, for $\nabla \cdot \vec{B} = 0 \Rightarrow \underline{\vec{\beta} \cdot \vec{H} = 0} \quad (11)$

These four equations (8)-(11) are very important in electromagnetics. We will use them extensively.

These equations also provide valuable information for the general behavior of plane waves in a lossless, simple material.

Eqs. (8) & (9) tell us that $\vec{E} \perp \vec{H}$, while (10) & (11) tell us that \vec{E} & \vec{H} are both \perp to the direction of propagation.



Dispersion Relationship

Taking $(\vec{\beta} \times \vec{B})$ we find

$$\vec{\beta} \times (\vec{\beta} \times \vec{E}) = \pm \omega \mu \vec{\beta} \times \vec{H} \quad (12)$$

substituting (9) in the RHS

$$\vec{\beta} \times (\vec{\beta} \times \vec{E}) = \pm \omega \mu (\mp \omega \epsilon \vec{E}) = -\omega^2 \mu \epsilon \vec{E}$$

or

$$\vec{\beta} \times (\vec{\beta} \times \vec{E}) + \beta^2 \vec{E} = 0 \quad (13)$$

We'll apply the vector id. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$ to the LHS of (13) giving

$$(\vec{\beta} \cdot \vec{E}) \vec{\beta} - (\vec{\beta} \cdot \vec{\beta}) \vec{E} + \beta^2 \vec{E} = 0$$

From (10), $\vec{\beta} \cdot \vec{E} = 0$ so that

$$(\vec{\beta} \cdot \vec{\beta} - \beta^2) \vec{E} = 0. \quad (14)$$

A non-trivial solution requires that

$$\underline{\vec{\beta} \cdot \vec{\beta} = \beta^2} \quad (15)$$

This is the dispersion relation for plane waves propagating in simple material. In cartesian coords. where

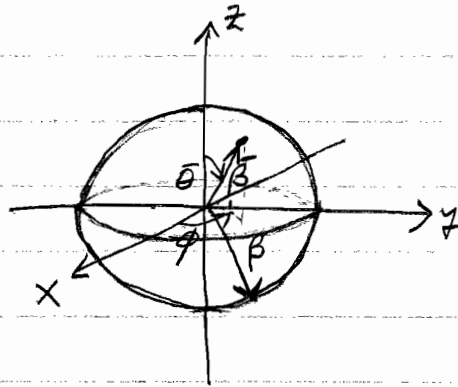
$$\vec{\beta} = \bar{a}_x \beta_x + \bar{a}_y \beta_y + \bar{a}_z \beta_z$$

Then,

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$$

(16)

This is an equation for a sphere. (15) states that $\vec{\beta}$ lies on the surface of a sphere:



As the direction of propagation changes, the components of β change, but the tip of $\vec{\beta}$ remains on the sphere.

A material in which the dispersion surface is a sphere is an isotropic space. From (15) or (16) we have one scalar equation that allows us to find one of the three comp's of $\vec{\beta}$. The other two require more information from the problem such as source conditions or boundary conditions.

In the case of the spherical coordinates θ & ϕ shown above, referring to eqn. (II-13b)

$$\beta_x = \beta \sin \theta \cos \phi$$

$$\beta_y = \beta \sin \theta \sin \phi$$

and
$$\beta_z = \beta \cos \theta$$