Lecture 4 — Transverse Electromagnetic Waves. Complex Poynting Vector.

Our first application of the wave equation and its solution in Cartesian coordinates we derived in the preceding lecture is plane waves propagating in unbounded space.

We'll consider a rather simplistic and contrived situation to start this discussion, but it will be a good example to begin with. To keep things simple, let's imagine we have only an x component to $\bar{E}$ and only z variation.

In a lossless, source-free space as we learned in the previous lecture

$$\nabla^2 \bar{E} + \beta^2 \bar{E} = 0 \qquad (3\text{-}20),(1)$$

$$\nabla^2 \bar{H} + \beta^2 \bar{H} = 0 \qquad (2)$$

which in Cartesian coords, the Ex field solution is

$$E_x(x,y,z) = \left(A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x}\right)\left(A_2 e^{-j\beta_y y} + B_2 e^{+j\beta_y y}\right)\left(A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right) \qquad (3)$$

where

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 \left(= \omega^2 \mu \epsilon\right) \qquad (4)$$

In the case of z variation only, (3) becomes

$$E_x(z) = E_0^+ e^{-j\beta_z z} + E_0^- e^{+j\beta_z z}$$

but since $\beta_z = \beta$ now, then

$$E_x(z) = E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z} \equiv E_x^+(z) + E_x^-(z) \qquad (4\text{-}2),(5)$$

These terms are the phasor representation of waves travelling in the +z and -z directions, respectively. To see this, we'll convert the second term in (5) to the time domain:

$$\mathcal{E}_x^-(z,t) = Re\left[E_o^- e^{+j\beta z} e^{j\omega t}\right] = Re\left[|E_o^-| e^{j\phi^+} e^{j(\omega t+\beta z)}\right]$$

where we're defining $E_o^- = |E_o^-| e^{j\phi^+}$. Continuing,

$$\mathcal{E}_x^-(z,t) = |E_o^-| \cos(\omega t + \beta z + \phi^+) = |E_o^-| \cos\left[\omega\left(t + \frac{\beta}{\omega} z\right) + \phi^+\right] \qquad (6)$$

Any suitably differentiable fct. that has t, ω, & β appearing exclusively in the form as $t + \frac{\beta}{\omega} z$ is a wave propagating in the $-z$ direction with speed

$$v_p \equiv \frac{\omega}{\beta} \qquad (4\text{-}7),\ (7)$$

This wave also has a magnetic field associated with it, which we can determine using Maxwell's equations. From Faraday's law in point form

$$\nabla \times \bar{E} = -j\omega\mu \bar{H}$$

or

$$\bar{H} = \frac{j}{\omega\mu} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \frac{j}{\omega\mu} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix}$$

$$= \frac{j}{\omega\mu}(-\hat{y})\left(-\frac{\partial E_x}{\partial z}\right) = \hat{y}\,\frac{j}{\omega\mu}\,\frac{\partial E_x}{\partial z} \qquad (8)$$

Substituting (5) in (8) leaves

$$\bar{H} = \hat{y}\,\frac{j}{\omega\mu}\left[(-j\beta)E_0^+ e^{-j\beta z} + (j\beta)E_0^- e^{+j\beta z}\right]$$

$$= \hat{y}\,\frac{\beta}{\omega\mu}\left(E_0^+ e^{-j\beta z} - E_0^- e^{+j\beta z}\right) \equiv \hat{y}\left(H_y^+ - H_y^-\right) \qquad (9)$$

The quantity

$$\frac{\omega\mu}{\beta} = \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}} \equiv \eta \qquad (10)$$

has units of impedance and is called the intrinsic impedance of the material.

There are a few very important points to notice in (5) and (9):

- $\bar{E} \perp \bar{H}$
- Both $\bar{E}$ & $\bar{H}$ are $\perp$ to $\pm z$, the directions of propagation.
- The direction of the cross product $\bar{E} \times \bar{H}$ is also the direction of propagation
- In planes $\perp$ to the directions of propagation ($\pm z$), $\bar{E}$ & $\bar{H}$ are constant (in both magnitude & phase). In other words, the fields are uniform.

Because of this latter property, these types of EM fields are called uniform plane waves (UPWs). Further, because $\bar{E}$ & $\bar{H}$ are $\perp$ to the direction of propagation, they are called Transverse Electric and Magnetic (TEM) waves.

- For the $+z$ propagating wave, the ratio of E_x to H_y is $E_x^+/H_y^+ = \eta$ (11) while for the $-z$ prop. wave, $E_x^-/H_y^- = -\eta$.

- The wavelength of the wave is

$$\lambda = \frac{2\pi}{\beta}$$

- the wave speed is

$$v_p = \frac{1}{\sqrt{\mu\epsilon}}$$

Power Flow and Poynting Vector

The Poynting vector we defined in lecture 2

$$\bar{\mathcal{P}}(\bar{r}) = \bar{\mathcal{E}} \times \bar{\mathcal{H}} \qquad (12)$$

can be calculated here for the $+z$ & $-z$ prop. waves. We need the time-domain form of the fields in (5) & (9) to calculate (12). For example, assuming E_0^+ is a real number:

$$\bar{\mathcal{P}}^+ = \bar{\mathcal{E}}^+ \times \bar{\mathcal{H}}^+ = \hat{x}\, E_0^+ \cos(\omega t - \beta z) \times \left[\hat{y}\, \frac{E_0^+}{\eta} \cos(\omega t - \beta z)\right]$$

$$= \hat{z}\, \frac{(E_0^+)^2}{\eta} \cos^2(\omega t - \beta z) \quad \frac{W}{m^2}. \qquad (4.8c), (13)$$

Notice that the power flow is in the $+z$ direction, the same direction as the wave propagates.

The time average of this power flow is

$$\langle \bar{\mathcal{S}}^+ \rangle \equiv \frac{1}{T}\int_{t_o}^{t_o+T} \bar{\mathcal{S}}(t)\,dt = \frac{1}{T}\int_{t_o} \hat{z}\,\frac{(E_o^+)^2}{\eta}\cos^2(\omega t-\beta z)\,dt$$

using the trig i.d. $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$

$$\langle \bar{\mathcal{S}}^+ \rangle = \hat{z}\,\frac{(E_o^+)^2}{\eta}\cdot\frac{1}{T}\int_{t_o}^{t_o+T}\frac{1}{2}\Big[1+\underbrace{\cos(2\omega t-2\beta z)}_{\text{time average}=0}\Big]\,dt$$

$$= \hat{z}\,\frac{(E_o^+)^2}{2\eta} \quad \left[\frac{W}{m^2}\right] \qquad (14)$$

Complex Poynting Vector

There is another way to calculate this time average value that can be done strictly from the phasor form of the fields. In terms of phasors, we can write $\bar{\mathcal{E}}$ & $\bar{\mathcal{H}}$ in the following forms

$$\bar{\mathcal{E}}(\bar{r},t) = \mathcal{R}e\left[\bar{E}(\bar{r})e^{j\omega t}\right] = \frac{1}{2}\left\{\bar{E}(\bar{r})e^{j\omega t} + \left[\bar{E}(\bar{r})e^{j\omega t}\right]^*\right\} \quad (1\text{-}67a),(15)$$

$$\bar{\mathcal{H}}(\bar{r},t) = \mathcal{R}e\left[\bar{H}(\bar{r})e^{j\omega t}\right] = \frac{1}{2}\left\{\bar{H}(\bar{r})e^{j\omega t} + \left[\bar{H}(\bar{r})e^{j\omega t}\right]^*\right\} \quad (1\text{-}67b),(16)$$

where we have made use of the fact that for a complex number $c = a + jb$, $\mathcal{R}e(c) = \frac{1}{2}(c + c^*)$.

Substituting (15) & (16) into (12)

$$\bar{\mathcal{S}}(\bar{r}) = \frac{1}{2}\left(\bar{E}e^{j\omega t} + \bar{E}^* e^{-j\omega t}\right) \times \frac{1}{2}\left(\bar{H}e^{j\omega t} + \bar{H}^* e^{-j\omega t}\right)$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(\bar{E}\cancel{e^{j\omega t}} \times \bar{H}^* \cancel{e^{-j\omega t}} + \bar{E}^* \cancel{e^{-j\omega t}} \times \bar{H} e^{j\omega t}\right) + \right.$$
$$\left.\frac{1}{2}\left(\bar{E}\times\bar{H} e^{j2\omega t} + \bar{E}^* \times \bar{H}^* e^{-j2\omega t}\right)\right]$$

or

$$\bar{\mathcal{S}}(\bar{r}) = \frac{1}{2}\left\{\frac{1}{2}\left[\bar{E}\times\bar{H}^* + (\bar{E}\times\bar{H}^*)^*\right] + \frac{1}{2}\left[\bar{E}\times\bar{H} e^{j2\omega t} + (\bar{E}\times\bar{H} e^{j2\omega t})^*\right]\right\}$$

$\therefore$

$$\bar{\mathcal{S}}(\bar{r}) = \frac{1}{2}\left[\mathrm{Re}\left(\bar{E}\times\bar{H}^*\right) + \mathrm{Re}\left(\bar{E}\times\bar{H} e^{j2\omega t}\right)\right] \qquad (1\text{-}69), (17)$$

Since the first term is not a fct. of time and the second is harmonic with time (having zero time avg.), we see easily from (17) that

$$\langle \mathcal{S}(\bar{r}) \rangle = \frac{1}{2}\mathrm{Re}\left(\bar{E}\times\bar{H}^*\right) \qquad (18)$$

If we define a complex Poynting vector as

$$\bar{S}(\bar{r}) = \bar{E}(\bar{r}) \times \bar{H}^*(\bar{r}) \qquad (19)$$

then $\frac{1}{2}\mathrm{Re}[\bar{S}] = \langle \bar{\mathcal{S}} \rangle$. Note that your text actually defines $\bar{S} = \langle \mathcal{S} \rangle$ as in (1-70), which is not standard.

For the UPW example begun at the beginning of this lecture, we can use the phasor $\bar{E}$ & $\bar{H}$ of (5) & (9) to give from (19):

$$\bar{S}^+ = \left(\hat{x}\, E_o^+ e^{-j\beta z}\right) \times \left(\hat{y}\, \frac{E_o^+}{\eta} e^{-j\beta z}\right)^*$$

if η is real, then

$$\bar{S}^+ = \hat{z}\, \frac{|E_o^+|^2}{\eta} \qquad (20)$$

From (18),

$$\langle \mathcal{S} \rangle = \frac{1}{2} \mathrm{Re}(\bar{S}) \quad \text{or}$$

$$\langle \mathcal{S}^+ \rangle = \frac{1}{2} \mathrm{Re}(\bar{S}^+) = \hat{z}\, \frac{|E_o^+|^2}{2\eta} \qquad (21)$$

This agrees (14), which was derived exclusively from the time domain form of the fields, provided E_o^+ is real, as assumed earlier.