EE 692A EE
Whites

Lecture 31 — Solutions to Characteristic Equations. Cutoff Condition & Field Patterns

The remaining task for completing this solution to the grounded dielectric slab wvgd is to find the axial wavenumber of the propagating modes.

We will present a graphical solution that provides tremendous physical insight, as well as reasonably accurate values. Numerical root-finding is best for accuracy, but provides no physical insight.

The overall philosophy of the graphical method is to plot two slowly varying fcts. in the same plane & look for intersections. These are solutions that simultaneously satisfy both equations.

Both the TE^z & TM^z modes satisfy the same dispersion relations:

$$0 \le x \le d \;:\quad \beta_{x1}^2 + \beta_z^2 = \omega^2 \mu_0 \epsilon_1 \qquad (1)$$

$$d \le x \;:\quad \beta_z^2 - \alpha_{x2}^2 = \omega^2 \mu_0 \epsilon_2 \qquad (2)$$

Subtracting (2) from (1) we have

$$\beta_{x1}^2 + \alpha_{x2}^2 = \beta_0^2 \left(\epsilon_{r1} - \epsilon_{r2}\right) \qquad (3)$$

multiplying (3) by d^2 gives

$$\boxed{(\beta_{x1} d)^2 + (\alpha_{x2} d)^2 = (\beta_0 d)^2 \left(\epsilon_{r1} - \epsilon_{r2}\right)} \qquad (4)$$

This is an equation of a circle w/ x-axis $\beta_{x1}d$, y-axis $\alpha_{x2}d$ centered at the origin w/ radius

$$R = \beta_0 d\sqrt{\epsilon_{r1} - \epsilon_{r2}} \tag{5}$$

Equation (4) is one curve. The second curve will come from the characteristic equation which is different for TE^z and TM^z modes.

- TE^z modes: Beginning w/ the characteristic eqn

$$\tan(\beta_{x1}d) = -\frac{\beta_{x1}}{\alpha_{x2}}$$

multiply by d & rearrange to give

$$\boxed{\alpha_{x2}d = -\beta_{x1}d\ \cot(\beta_{x1}d)} \tag{6}$$

- TM^z modes: Begin w/ characteristic eqn

$$\tan(\beta_{x1}d) = \frac{\epsilon_1 \alpha_{x2}}{\epsilon_2 \beta_{x1}}$$

multiplying by d and rearrange to give

$$\boxed{\alpha_{x2}d = \frac{\epsilon_2}{\epsilon_1}\beta_{x1}d\tan(\beta_{x1}d)} \tag{7}$$

The graphical solution is obtained by plotting (4) in the $\beta_{x1}d - \alpha_{x2}d$ plane along w/ either (6) or (7). The intersections of the two curves represents a simultaneous solution to the two equations giving values for $\beta_{x1}d$ and $\alpha_{x2}d$.

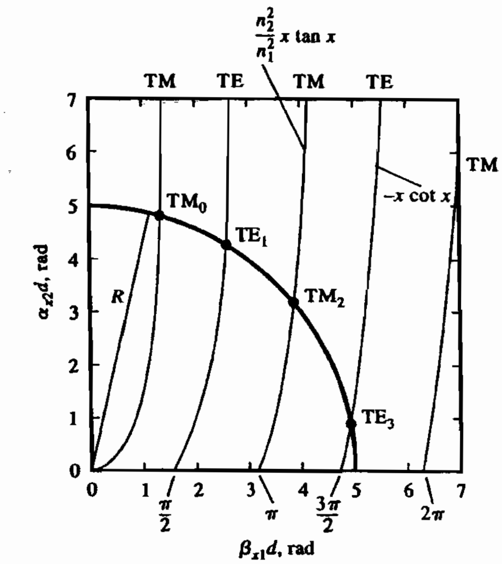


FIGURE 8.18
Graphical solution for the grounded dielectric slab waveguide. For the TM curves, $n_2/n_1 = 0.990$.

Each intersection represents a possible propagating mode. Once β_{x1} or α_{x2} are known, can determine β_z from dispersion relationships (1) or (2).

Physical Insights From Graphical Solution

As promised earlier, the graphical solution provides physical insights not apparent from a strictly numerical solution. These include:

1. From (5), R increases if: $f \uparrow$, $d \uparrow$ and/or $\epsilon_1 \uparrow$ or $\epsilon_2 \downarrow$

   As R increases, the more intersections will generally occur meaning more (possible) propagating modes.

2. Notice the TM_0 mode. When $R < \frac{\pi}{2}$ the only propagating mode is TM_0. Single mode operation. Additionally, no cutoff frequency.

3. At cutoff, we see from graphical solution that R will be integer multiple of $\frac{\pi}{2}$.

$$R = \frac{m\pi}{2} = \beta_{o,c}\, d\sqrt{\epsilon_{r1}-\epsilon_{r2}} \qquad (8)$$

s.t.

$$f_{c,m} = \frac{m}{4d\sqrt{\mu_o\epsilon_o}\sqrt{\epsilon_{r1}-\epsilon_{r2}}} \qquad (9)$$

are the cutoff frequencies for the modes where $m = 0, 2, 4, \ldots$ for TM_m modes while $m = 1, 3, 5, \ldots$ for TE_m modes.

4. Also at cutoff, we observe from the graphical solution that $\underline{\underline{\alpha_{x2} = 0}}$. (10)

Other that for TM_0, at cutoff $\beta_{x1} \neq 0$ which means that $\beta_z \neq 0$. This is <u>very</u> <u>different</u> behavior than hollow metallic waveguides where cutoff defined when $\beta_z = 0$.

Cutoff and Critical Angle

It is instructive to dig deeper into this cutoff condition. As we did w/ hollow metallic waveguides, we'll expand the field solutions into "bouncing" plane waves.

Considering TM modes, we found in the previous lecture:

$$H_{y1} = -\frac{j\omega\epsilon_1}{\beta_{x1}} C_5 \cos(\beta_{x1}x)\, e^{-j\beta_z z} \quad (11)$$

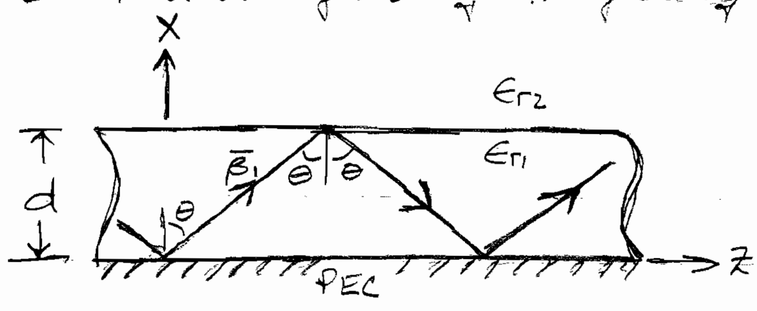
and

$$H_{y2} = -\frac{j\omega\epsilon_2}{\alpha_{x2}} C_5 \sin(\beta_{x1}d)\, e^{-\alpha_{x2}(x-d)}\, e^{-j\beta_z z} \quad (12)$$

Inside the slab, (11) represents upward & downward propagating UPWs when we expand the $\cos(\beta_{x1}x)$ term

$$H_{y1} = -\frac{j\omega\epsilon_1}{2\beta_{x1}} C_5 \left[e^{j(\beta_{x1}x - \beta_z z)} + e^{-j(\beta_{x1}x + \beta_z z)} \right] \quad (13)$$

If we consider a single ray trajectory as



Inside the slab,

$$\bar{\beta}_1 = \hat{x}\beta_{x1} + \hat{z}\beta_z$$
$$= \hat{x}\beta_1\cos\theta + \hat{z}\beta_1\sin\theta \quad (14)$$

As the frequency of operation changes, this θ changes – becoming larger as $f \uparrow$. Of particular interest is θ at cutoff, which we'll denote as θ_c.

From the dispersion relation in region 2

$$\beta_z^2 - \alpha_{x2}^2 = \omega^2 \mu_0 \epsilon_2$$

at cutoff $\alpha_{x2} = 0 \Rightarrow$

$$\beta_{z,c} = \beta_{0,c}\sqrt{\epsilon_{r2}} \qquad (15)$$

From (14), at cutoff $\beta_{z,c} = \beta_1 \sin\theta_c \qquad (16)$

Equating (15) & (16) we find

$$\beta_{1,c} \sin\theta_c = \beta_{0,c}\sqrt{\epsilon_{r2}}$$

or

$$\underline{\theta_c} = \sin^{-1}\left(\frac{\beta_{0,c}\sqrt{\epsilon_{r2}}}{\beta_{1,c}}\right) = \underline{\sin^{-1}\left(\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}\right)} \qquad (17)$$

Same as the critical angle of incidence for UPW incident on a half space. No coincidence, this is a clue as to how such a waveguide operates – through total internal reflection @ upper interface.

- $f < f_c$: $\theta < \theta_c$ & no internal reflection.
- $f > f_c$: $\theta > \theta_c$ & total internal reflection.

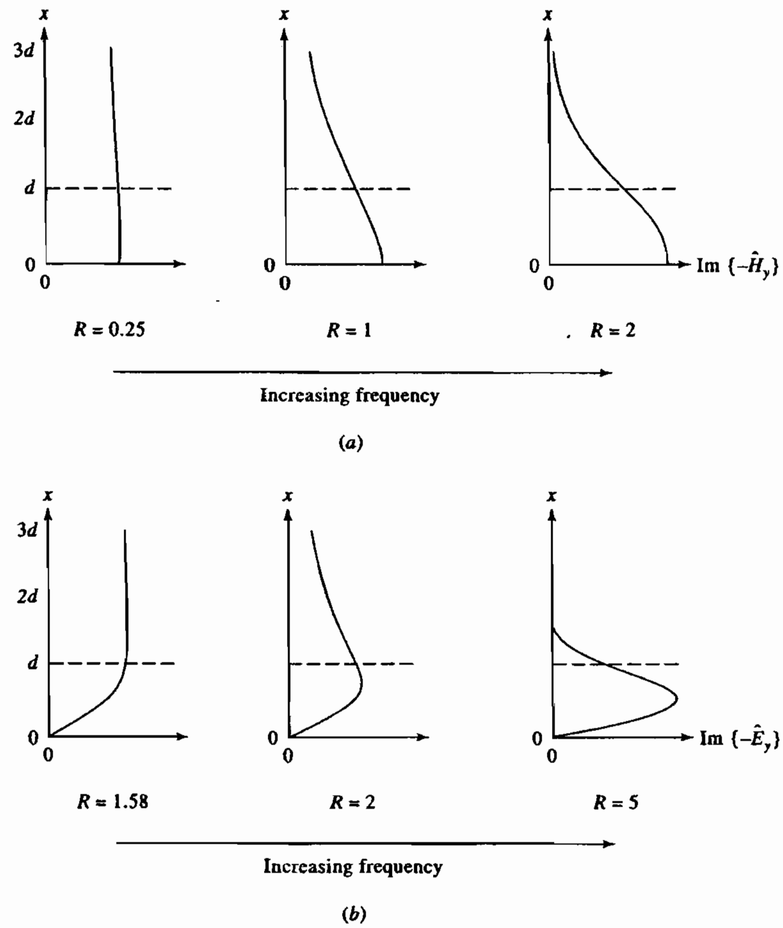


FIGURE 8.21
Plots of the y-component fields in the grounded dielectric slab waveguide of Fig. 8.17: (*a*) H_y for the TM_0 mode with $n_2/n_1 = 0.833$; (*b*) E_y for the TE_1 mode.

Source: C. R. Paul, K. W. Whites and S. A. Nasar, *Introduction to Electromagnetic Fields*. New York: McGraw-Hill, third ed., 1998.

Example N31.1

Grounded dielectric slab is GaAs ($\epsilon_{r,1}=12.96$) with GaAlAs cover ($\epsilon_{r2}=12.70$). Excited by near-infrared laser source w/ $\lambda_0=0.82\,\mu m$. Find max d for only single mode operation. For this d, compute β_{x1}, α_{x2}, β_z & ray angle θ.

Use previous figure for graphical solution since

$$\frac{n_2}{n_1} = \frac{\sqrt{12.70}}{\sqrt{12.96}} = 0.990 .$$

Single mode operation up to $R = \frac{\pi}{2}$ (only TM_0 mode propagates). From (5)

$$R = \frac{\pi}{2} = \frac{2\pi}{\lambda_0} d \sqrt{12.96-12.70}$$

$$\Rightarrow \qquad \underline{d = 0.404\,\mu m}$$

For single mode operation, slab thickness less than approx. one-half source wavelength so this "slab" is very thin film.

Construct circle on Fig 8.18 w/ radius $\frac{\pi}{2}$. Intersects TM_0 curve at

$$\beta_{x1} d \approx 0.95 \text{ rad} \Rightarrow \beta_{x1} \approx 2.35 \frac{\text{rad}}{\mu m}$$

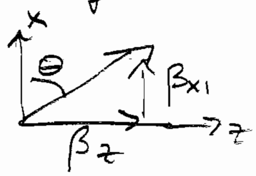
& $$\alpha_{x2} d \approx 1.25 \text{ rad} \Rightarrow \alpha_{x2} \approx 3.09 \frac{\text{rad}}{\mu m}$$

Using dispersion relation (1) in the slab, we find that

$$\beta_z \approx 27.5 \ \frac{\text{rad}}{\mu m}$$

(Root finding gives $\beta_{x_1} = 2.327 \frac{\text{rad}}{\mu m}$, $\alpha_{x_2} = 3.119 \frac{\text{rad}}{\mu m}$; $\beta_z = 27.49 \frac{\text{rad}}{\mu m}$)

Ray angle is



$$\theta = \tan^{-1}\left(\frac{\beta_z}{\beta_{x_1}}\right) = 85.2^\circ$$

Very "steep" angle, but θ_c is very large:

$$\theta_c = \sin^{-1}\left(\sqrt{\frac{12.70}{12.96}}\right) = 81.9^\circ$$