Lecture 3 – Solutions to Wave Equation in Cartesian Coords.

EM sources radiate signals in the form of wave disturbances in the EM field. In this lecture, we will derive the EM wave eqn in coordinate-free form. Then we will determine solutions in the Cartesian coord. system. (a) Virtually all our work in this course is w/ sinusoidal steady state. So, we'll start the derivation of the EM wave eqns. w/ Maxwell's curl equations in vector phasor form ($e^{j\omega t}$ dependence) in a simple material:

$$\nabla \times \bar{E} = -j\omega \bar{B} - \bar{M}_i = -j\omega\mu \bar{H} - \bar{M}_i \tag{1}$$

$\bar{J}_i$ & $\bar{M}_i$ are impressed sources. →

$$\nabla \times \bar{H} = \bar{J}_c + j\omega \bar{D} + \bar{J}_i = \sigma \bar{E} + j\omega\epsilon \bar{E} + \bar{J}_i \tag{2}$$

Next, we take the curl of (1)

$$\nabla \times \nabla \times \bar{E} = -j\omega\mu \nabla \times \bar{H} - \nabla \times \bar{M}_i \tag{3}$$

and substitute (2) in (3)

$$\nabla \times \nabla \times \bar{E} = -j\omega\mu \left(\sigma \bar{E} + j\omega\epsilon \bar{E} + \bar{J}_i \right) - \nabla \times \bar{M}_i \tag{4}$$

Using the vector id. $\nabla \times \nabla \times \bar{E} = \nabla(\nabla \cdot \bar{E}) - \underbrace{\nabla^2 \bar{E}}_{\text{careful!}}$

Eqn. (4) becomes

$$\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -j\omega\mu\sigma \bar{E} + \omega^2 \mu\epsilon \bar{E} - j\omega\mu \bar{J}_i - \nabla \times \bar{M}_i \tag{5}$$

Now, from Gauss's law in a simple material

$$\nabla \cdot \bar{D} = \rho_{ve} \Rightarrow \nabla \cdot \bar{E} = \frac{\rho_{ve}}{\epsilon} \tag{6}$$

and substituting into (5)

$$\nabla\left(\frac{\rho_{ve}}{\epsilon}\right) - \nabla^2 \bar{E} - \omega^2\mu\epsilon \bar{E} = -j\omega\mu\sigma \bar{E} - j\omega\mu \bar{J}_i - \nabla \times \bar{M}_i \tag{7}$$

This is the vector phasor wave equation for the electric field.

Rearranging, (6) becomes

$$\nabla^2 \bar{E} + (\omega^2\mu\epsilon - j\omega\mu\sigma)\bar{E} = \frac{1}{\epsilon}\nabla\rho_{ve} + j\omega\mu\bar{J}_i + \nabla\times\bar{m}_i \qquad (3\text{-}16a),(8)$$

We'll define $\gamma^2 \equiv j\omega\mu\sigma - \omega^2\mu\epsilon \;\; [m^{-1}]$ (3-17c),(9)
as the propagation constant. This is a complex number

$$\gamma = \alpha + j\beta \qquad (3\text{-}17d),(10)$$

where

$\alpha \hat{=}$ attenuation constant [Np/m] (3-17e),(11)
$\hat{=} Re(\gamma)$

$\beta \hat{=}$ phase constant [rad/m] (3-17f),(12)
$= Im(\gamma)$

(Notice that $\beta^2 \neq \omega^2\mu\epsilon$!)

A similar derivation leads to vector phasor wave eqn for the magnetic field:

$$\nabla^2 \bar{H} + (\omega^2\mu\epsilon - j\omega\mu\sigma)\bar{H} = \frac{1}{\mu}\nabla\rho_{vm} + j\omega\epsilon\bar{m}_i + \sigma\bar{m}_i - \nabla\times\bar{J}_i \qquad (3\text{-}16b),(13)$$

Without the fictitious magnetic sources, these wave equations (8) & (13) become:

$$\nabla^2\bar{E} - \gamma^2\bar{E} = \frac{1}{\epsilon}\nabla\rho_{ve} + j\omega\mu\bar{J}_i \qquad (14)$$

&

$$\nabla^2\bar{H} - \gamma^2\bar{H} = -\nabla\times\bar{J}_i \qquad (15)$$

Lastly, in a source free & lossless space, the wave eqns reduce to

$$\nabla^2\bar{E} + \beta^2\bar{E} = 0 \qquad (3\text{-}18a),(16)$$

&

$$\nabla^2\bar{H} + \beta^2\bar{H} = 0 \qquad (3\text{-}18b),(17)$$

$$\text{where } \beta^2 \equiv \omega^2\mu\epsilon \qquad (3\text{-}18c),(18)$$

β is also called the wavenumber, & is sometimes referred to by the variable k.

Solutions to wave Eqn. in Cartesian Coord. System

One of the fantastic attributes of vector calculus, and in particular the wave equations in (8), (13)-(17), is they are valid for any coordinate system. That is, they are coordinate free representations of these equations.

Most often, an analytical sol'n to these wave equations requires a judicious choice of a coordinate system.

Determining the solutions to these partial differential equations is non-trivial. The separation of variables technique is one of only a few methods for determining such solutions. There are 11 coordinate systems for which separable solutions to the scalar wave equation is possible*: in three dimensions

$$(\hat{\nabla}^2\psi+\beta^2\psi=0)$$

1. Rectangular
2. Circular Cylinder
3. Elliptic Cylinder
4. Parabolic Cylinder
5. Spherical
6. Conical
7. Parabolic
8. Prolate Spheroidal
9. Oblate Spheroidal
10. Ellipsoidal
11. Paraboloidal

* Ref: P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," McGraw-Hill, 1953, Ch. 5.

(In this course) We'll look at solutions only in the cartesian coordinate system. There are plenty of things to learn about in electromagnetics using just this coordinate system!

Source - Free & Lossless Media

For source free & lossless media, the phasor-domain vector wave equation for $\bar{E}$ is given in (16)

$$\nabla^2 \bar{E}(x,y,z) + \beta^2 \bar{E}(x,y,z) = 0 \qquad (3\text{-}20), (19)$$

This is actually a compact way of expressing three scalar equations

$$\nabla^2 E_x + \beta^2 E_x = 0 \qquad (3\text{-}20a), (20)$$
$$\nabla^2 E_y + \beta^2 E_y = 0 \qquad (3\text{-}20b), (21)$$
$$\nabla^2 E_z + \beta^2 E_z = 0 \qquad (3\text{-}20c), (22)$$

All three have the same form, so a general solution to one is a solution for all.

We will use separation of variables to determine solutions to these eqns. This begins by assuming a product soln of the form:

$$E_x(x,y,z) = f(x) \cdot g(y) \cdot h(z) \qquad (3\text{-}23), (23)$$

Notice that each of the unknown functions f, g & h are dependent on one coord. variable only!

Sub. (23) → (20), dividing by $f \cdot g \cdot h$ & rearranging yields

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = -\beta^2 \qquad (3\text{-}25a), (24)$$

The second & third terms are not fts of x, and neither is β^2. Consequently, in order for (24) to remain valid $\forall x$ requires that the first term is also not a ft. of x! Does this mean f is not a fct of x? No.

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\beta_x^2 \quad \text{or} \quad \frac{d^2 f}{dx^2} + \beta_x^2 f = 0 \qquad (3\text{-}26a), (25)$$

A similar argument leads to

$$\frac{d^2 g}{dy^2} + \beta_y^2 g = 0 \qquad (3\text{-}26b),(26)$$

and

$$\frac{d^2 h}{dz^2} + \beta_z^2 h = 0 \qquad (3\text{-}26c),(27)$$

Furthermore $\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$ (3-27),(28)
in order for (24) to be satisfied.

Eqns. (25)-(27) are harmonic equations and you are well familiar with their solutions. For example,

$$f_1(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x} \quad \text{traveling wave} \quad (3\text{-}28a),(29)$$

or

$$f_2(x) = C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x) \quad \text{standing wave} \quad (3\text{-}28b),(30)$$

Similar solutions for $g(y)$ & $h(z)$ can be written, as in the text.

The total solution for E_x from (23) is the product of these solutions for f, g & h. For purely traveling waves, as an example,

$$E_x(x,y,z) = \left[A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x}\right] \cdot \left[A_2 e^{-j\beta_y y} + B_2 e^{+j\beta_y y}\right] \cdot \left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right] \quad (31)$$

A similar process can be used to develop the wave solutions for the E_y & E_z field components in (21) & (22).

Source Free & Lossy Media

The source-free wave equations in this case are from (16) & (17):

$$\nabla^2 \bar{E} - \gamma^2 \bar{E} = 0 \qquad (3\text{-}37),(32)$$

&

$$\nabla^2 \bar{H} - \gamma^2 \bar{H} = 0 \qquad (33)$$

Solutions to these in Cartesian components are determined by the separation of variables, as before. As shown in the text, for E_x

$$E_x(x,y,z) = f(x)\cdot g(y)\cdot h(z) \qquad (3\text{-}39), (34)$$

possible solutions for $f(x)$ are

$$f_1(x) = A_1 e^{-\gamma_x x} + B_1 e^{+\gamma_x x} \qquad (3\text{-}40a), (35)$$

and $\quad f_2(x) = C_1 \cosh(\gamma_x x) + D_1 \sinh(\gamma_x x) \qquad (3\text{-}40b), (36)$

where $\quad \gamma_x = \alpha_x + j\beta_x \qquad (3\text{-}46), (37)$

Similar solutions for $g(y)$ & $h(z)$ are obtained also in order to satisfy (32),

$$\gamma_x^2 + \gamma_y^2 + \gamma_z^2 = \gamma^2 \qquad (3\text{-}43), (\cancel{37})\ 38.$$

A propagating & attenuating wave solution to (37) would have the form

$$E_x(x,y,z) = \left[A_1 e^{-\gamma_x x} + B_1 e^{+\gamma_x x}\right]\cdot\left[A_2 e^{-\gamma_y y} + B_2 e^{+\gamma_y y}\right]\cdot\left[A_3 e^{-\gamma_z z} + B_3 e^{+\gamma_z z}\right] \qquad (\cancel{38})\ 39.$$

We will make extensive use of these general solutions to the wave equation in lossless & lossy media over much of the semester in our study of plane waves, & waveguides.