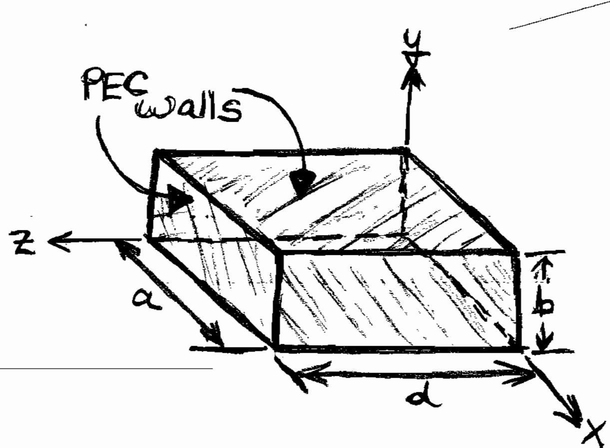
Rectangular Waveguide Resonant Cavities

Resonant devices with extremely high Q can be realized by placing metal end caps onto a rectangular waveguide.



Energy is coupled into the cavity through a small hole (or holes) on a side wall, or by a small wire or loop extended through a side wall. We'll assume the excitation occurs on a z=constant side wall.

A wave excited at $z=0$, for example, may prop. as a waveguide mode in $+z$, reflect off the wall at $z=d$, then prop. back to the $z=0$ wall.

At certain frequencies, these incident and reflected waves will add together in phase, strengthening the oscillations. Other than at these special frequencies, the waves interfere destructively inside the cavity so no energy can be coupled into the cavity.

We can determine these special frequencies by applying b.c.'s at $z=0$ & $z=d$.

We must be a bit careful here to apply the b.c.'s to the total E_{tan} at these surfaces.

- TE^z cavity modes. With propagation in the $\pm z$ directions, then

For $+z$ prop:
$$H_z^+ = A_{mn} \cos_x^a \cos_y^b e^{-j\beta_z z} \tag{1}$$

$$E_x^+ = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} A_{mn} \cos_x^a \sin_y^b e^{-j\beta_z z} \tag{2}$$

$$E_y^+ = \frac{-j\omega\mu\beta_{xm}}{\beta_{cmn}^2} A_{mn} \sin_x^a \cos_y^b e^{-j\beta_z z} \tag{3}$$

For prop. in the $-z$ direction, $\underline{\beta_z \to -\beta_z}$ in (1)–(3) gives

$$H_z^- = B_{mn} \cos_x^a \cos_y^b e^{+j\beta_z z} \tag{4}$$

$$E_x^- = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} B_{mn} \cos_x^a \sin_y^b e^{+j\beta_z z} \tag{5}$$

$$E_y^- = \frac{-j\omega\mu\beta_{xm}}{\beta_{cmn}^2} B_{mn} \sin_x^a \cos_y^b e^{+j\beta_z z} \tag{6}$$

Using (2) & (5), the total $E_x^t = E_x^+ + E_x^-$ for mode mn is

$$E_x^t = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} \cos_x^a \sin_y^b \left(A_{mn} e^{-j\beta_z z} + B_{mn} e^{+j\beta_z z}\right) \tag{7}$$

E_x^t already satisfies b.c.'s at $x=0, a$ and $y=0, b$: That's how we originally determined $\beta_{xm} = \frac{m\pi}{a}$ & $\beta_{yn} = \frac{n\pi}{b}$.

Now need to apply boundary conditions at $z=0$ & d for $E_{tan}=0$.

$\cos\left(\frac{m\pi x}{a}\right)\ \sin\left(\frac{n\pi y}{b}\right)$

at $z=0$

Using (7). For $E_x^t = 0 = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} \cos_x^a \sin_y^b (A_{mn} + B_{mn})$

$$\Rightarrow \quad \underline{B_{mn} = -A_{mn}} \tag{8}$$

Sub. (8) into (7):

$\cos_x^a \sin_y^b$

$$E_x^t = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} A_{mn} \underbrace{\left(e^{-j\beta_z z} - e^{+j\beta_z z}\right)}_{= -j2\sin(\beta_z z)}$$

or

$$E_x^t = \frac{2\omega\mu\beta_{yn}}{\beta_{cmn}^2} A_{mn} \cos_x^a \sin_y^b \sin(\beta_z z) \tag{9}$$

At $z=d$, $E_x^t = 0 \underset{(9)}{=} \frac{2\omega\mu\beta_{yn}}{\beta_{cmn}^2} A_{mn} \cos_x^a \sin_y^b \sin(\beta_z d)$

$$\Rightarrow \quad \underline{\beta_z = \beta_{zp} = \frac{p\pi}{d}} \qquad p = 1, 2, 3, \cdots \tag{10}$$

It can be shown that this same condition (10) will also enforce the b.c. $E_y = 0$ at $z=0, d$ since using (3) & (6)

$$E_y^t = E_y^+ + E_y^- = \frac{-2\omega\mu\beta_{xm}}{\beta_{cmn}^2} A_{mn} \sin_x^a \cos_y^b \sin(\beta_z z) \tag{11}$$

With $\beta_z = \beta_{zp}$ in (10), E_y^t in (11) vanishes at $z=0, d$.

Notice that in (10), $p \neq 0$. If $p=0$ in (9) or (11), then $E_x^t = E_y^t = 0$ $\forall x, y, z$ inside the cavity. Because $E_z = 0$ for the TE^z modes, then $\bar{E} = 0$ if $p = 0$. With $\bar{E} = 0$, no resonance is possible. (Can also show that $H_z = 0$ for

$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$ $\qquad \beta_z = \sqrt{\beta^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$ assume any value.

$p = 0 \Rightarrow$ all TE^z fields vanish.)

Substituting (10) into the dispersion relation

$$\underbrace{\beta_{xm}^2 + \beta_{yn}^2 + \beta_{zp}^2} = \beta^2 = \omega^2 \mu \epsilon$$

then $$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 = \omega^2 \mu \epsilon \qquad (12).$$

The only degree of freedom left is the frequency of operation. From (12), the resonant frequencies of the cavity are thus

$$f_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2} \checkmark \qquad (13)$$

for TE^z_{mnp} modes where $\checkmark$ $m, n = 0, 1, 2, \cdots$ $(m = n \neq 0)$, &
$\checkmark$ $p = 1, 2, 3, \cdots$

- <u>TM^z cavity modes</u>. Referencing Pozar, 3rd edition, Ch. 3, equs. (3.100) and (3.101), for prop. in the $+z$ direction:

$$E_z^+ = A_{mn} \sin_x^a \sin_y^b e^{-j\beta_{z\cancel{mn}} z} \qquad (14)$$

and

$$E_x^+ = \frac{-j\beta_z}{\beta_c^2}\frac{\partial E_z}{\partial x} = \frac{-j\beta_{z\cancel{mn}}}{\beta_{cmn}^2} A_{mn} \beta_{xm} \cos_x^a \sin_y^b e^{-j\beta_{z\cancel{mn}} z} \qquad (15)$$

$$E_y^+ = \frac{-j\beta_z}{\beta_c^2}\frac{\partial E_z}{\partial y} = \frac{-j\beta_{z\cancel{mn}}}{\beta_{cmn}^2} A_{mn} \beta_{yn} \sin_x^a \cos_y^b e^{-j\beta_{z\cancel{mn}} z} \qquad (16)$$

For propagation in the $-z$ direction ($\beta_{z\cancel{mn}} \rightarrow -\beta_{z\cancel{mn}}$) then

$$E_z^- = B_{mn} \sin_x^a \sin_y^b e^{+j\beta_z z} \qquad (17)$$

and

$$E_x^- = \frac{+j\beta_z \beta_{xm}}{\beta_{cmn}^2} B_{mn} \cos_x^a \sin_y^b e^{+j\beta_{zmn} z} \qquad (18)$$

$$E_y^- = \frac{+j\beta_z \beta_{yn}}{\beta_{cmn}^2} B_{mn} \sin_x^a \cos_y^b e^{+j\beta_{zmn} z} \qquad (19)$$

Comparing (15), (16), (18), & (19) with the transverse $\bar{E}$ for TE^z modes in (2), (3), (5), & (6), we see that the transverse $\bar{E}$ for TE^z & TM^z modes have the same spatial dependencies. Consequently, applying boundary conditions at $z=0$ & d, we'll arrive at the same resonant frequency expression as in (13).

It turns out, though, that the index $p=0$ is allowed for TM^z modes. How can this be since $E_x = E_y = 0 \;\forall x,y,z$ when $p=0$? For TM^z modes, $E_z \neq 0$. From (14) and (17) with $B_{mn} = A_{mn}$:

$$E_z^t = E_z^+ + E_z^- = A_{mn} \sin_x^a \sin_y^b \left(e^{-j\beta_{zp} z} + e^{+j\beta_{zp} z}\right)$$

$$= 2 A_{mn} \sin_x^a \sin_y^b \cos(\beta_{zp} z) \qquad (20)$$

For $p=0$, $E_z^t = 2 A_{mn} \sin_x^a \sin_y^b \neq 0$.

So, for TM^z modes with $\underline{p=0}$, E_z^t is constant in z but varies in x & y.

What about $\bar{H}$? Can show that $H_x, H_y \propto \cos\left(\frac{p\pi}{d} z\right)$

such that $H_x, H_y \neq 0$. Consequently, $E_z, H_x,$ and H_y are non-zero so that TM^z_{mnp} modes are possible.

Consequently, for TM^z_{mnp} modes, the resonant frequencies are

$$f_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2} \tag{21}$$

where $m, n = 1, 2, 3, \ldots$;
$p = 0, 1, 2, \ldots$

Dominant Mode in a Rectangular Cavity Resonator

For the case when the cavity dimensions are related by $b < a < d$, as sketched on p. 1, (13) and (21) to determine that the cavity mode with the lowest resonant frequency is TE_{101}. This is the dominant or fundamental cavity mode.

The TE_{101} mode corresponds to a TE_{10} waveguide mode in a $\frac{\lambda_z}{2}$-long cavity.

Which mode resonates next as the frequency increases depends on the specific ratios of $\frac{b}{a}$ and $\frac{d}{a}$.

For example, in the case of a WR-90 waveguide that is 2-cm long in z:

Resonant frequency (GHz)	mode
9.959	TE_{101}
15.105	TE_{201}
~~16.146~~	~~TM_{110}~~
16.362	TE_{102}
16.549	TE_{011}
17.800	TE_{111}, TM_{111}
19.740	TM_{210}
19.917	TE_{202}

These are the resonant frequencies through 20 GHz. Notice:

- As the frequency increases, the "density" of resonant modes increases.
- At 17.800 GHz, both TE_{111} & TM_{111} can resonate. Two modes that have the same resonant frequency but different field patterns are called degenerate modes.

```
a = 0.9 * 2.54 / 100. ;
b = 0.4 * 2.54 / 100. ;
d = 0.02 ;
c0 = 2.998 * 10^8 ;
fres[m_, n_, p_] := c0 / (2. * Pi) * Sqrt[(m * Pi / a)^2 + (n * Pi / b)^2 + (p * Pi / d)^2]
```

TEz modes (where m and n can not simultaneously equal zero).

```
Table[{m, n, p, fres[m, n, p] / 10^9}, {m, 0, 4}, {n, 0, 4}, {p, 1, 4}] ;
Prepend[%, {"m", "n", "p", "fres (GHz)"}] ;
TableForm[%]
```

TE_{mnp}

Out[32]//TableForm=

m				n				p				fres (GHz)							
~~0~~	~~0~~	~~1~~	~~7.495~~	0	1	1	16.5485	0	2	1	30.4449	0	3	1	44.8919	0	4	1	59.4898
~~0~~	~~0~~	~~2~~	~~14.99~~	0	1	2	21.0328	0	2	2	33.0971	0	3	2	46.7312	0	4	2	60.8897
~~0~~	~~0~~	~~3~~	~~22.485~~	0	1	3	26.8934	0	2	3	37.0984	0	3	3	49.6456	0	4	3	63.154
~~0~~	~~0~~	~~4~~	~~29.98~~	0	1	4	33.4138	0	2	4	42.0656	0	3	4	53.4594	0	4	4	66.1941
1	0	1	9.95858	1	1	1	17.8003	1	2	1	31.143	1	3	1	45.3683	1	4	1	59.8501
1	0	2	16.3615	1	1	2	22.0313	1	2	2	33.7404	1	3	2	47.189	1	4	2	61.2418
1	0	3	23.4216	1	1	3	27.6813	1	2	3	37.6734	1	3	3	50.0768	1	4	3	63.4936
1	0	4	30.6887	1	1	4	34.0511	1	2	4	42.5736	1	3	4	53.8601	1	4	4	66.5181
2	0	1	15.1052	2	1	1	21.1151	2	2	1	33.1494	2	3	1	46.7683	2	4	1	60.9182
2	0	2	19.9172	2	1	2	24.7865	2	2	2	35.6007	2	3	2	48.5366	2	4	2	62.286
2	0	3	26.0301	2	1	3	29.9207	~~2~~	~~2~~	~~3~~	~~39.3482~~	2	3	3	51.3486	2	4	3	64.5014
2	0	4	32.723	2	1	4	35.8953	2	2	4	44.0625	2	3	4	55.0445	2	4	4	67.4808
3	0	1	21.0513	3	1	1	25.7068	3	2	1	36.2474	3	3	1	49.0129	3	4	1	62.6579
3	0	2	24.7323	3	1	2	28.7987	3	2	2	38.5019	3	3	2	50.703	3	4	2	63.9886
3	0	3	29.8757	3	1	3	33.3202	3	2	3	41.9914	3	3	3	53.401	3	4	3	66.1469
3	0	4	35.8578	3	1	4	38.7745	3	2	4	46.4381	3	3	4	56.964	3	4	4	69.0554
4	0	1	27.2791	4	1	1	31.0133	4	2	1	40.1853	4	3	1	51.9928	4	4	1	65.0154
4	~~0~~	~~2~~	~~30.2105~~	4	1	2	33.6207	4	2	2	42.2302	4	3	2	53.589	4	4	2	66.2988
4	0	3	34.5478	~~4~~	~~1~~	~~3~~	~~37.5663~~	4	2	3	45.4341	4	3	3	56.1485	4	4	3	68.3843
4	0	4	39.8343	4	1	4	42.4788	4	2	4	49.573	4	3	4	59.5473	4	4	4	71.2013

TMz modes

```
In[33]:= Table[{m, n, p, fres[m, n, p] / 10^9}, {m, 1, 4}, {n, 1, 4}, {p, 0, 4}] ;
         Prepend[%, {"m", "n", "p", "fres (GHz)"}] ;
         TableForm[%]
```

$\underline{TM_{mnp}}$

Out[35]//TableForm=

m				n				p				fres (GHz)			
1	1	0	16.1455	1	2	0	30.2277	1	3	0	44.7449	1	4	0	59.3789
1	1	1	17.8003	1	2	1	31.143	1	3	1	45.3683	1	4	1	59.8501
1	1	2	22.0313	1	2	2	33.7404	1	3	2	47.189	1	4	2	61.2418
1	1	3	27.6813	1	2	3	37.6734	1	3	3	50.0768	1	4	3	63.4936
1	1	4	34.0511	1	2	4	42.5736	1	3	4	53.8601	1	4	4	66.5181
2	1	0	19.7401	2	2	0	32.291	2	3	0	46.1638	2	4	0	60.4554
2	1	1	21.1151	2	2	1	33.1494	2	3	1	46.7683	2	4	1	60.9182
2	1	2	24.7865	2	2	2	35.6007	2	3	2	48.5366	2	4	2	62.286
2	1	3	29.9207	2	2	3	39.3482	2	3	3	51.3486	2	4	3	64.5014
2	1	4	35.8953	2	2	4	44.0625	2	3	4	55.0445	2	4	4	67.4808
3	1	0	24.5899	3	2	0	35.4641	3	3	0	48.4365	3	4	0	62.2081
3	1	1	25.7068	3	2	1	36.2474	3	3	1	49.0129	3	4	1	62.6579
3	1	2	28.7987	3	2	2	38.5019	3	3	2	50.703	3	4	2	63.9886
3	1	3	33.3202	3	2	3	41.9914	3	3	3	53.401	3	4	3	66.1469
3	1	4	38.7745	3	2	4	46.4381	3	3	4	56.964	3	4	4	69.0554
4	1	0	30.094	4	2	0	39.4802	4	3	0	51.4498	4	4	0	64.582
4	1	1	31.0133	4	2	1	40.1853	4	3	1	51.9928	4	4	1	65.0154
4	1	2	33.6207	4	2	2	42.2302	4	3	2	53.589	4	4	2	66.2988
4	1	3	37.5663	4	2	3	45.4341	4	3	3	56.1485	4	4	3	68.3843
4	1	4	42.4788	4	2	4	49.573	4	3	4	59.5473	4	4	4	71.2013