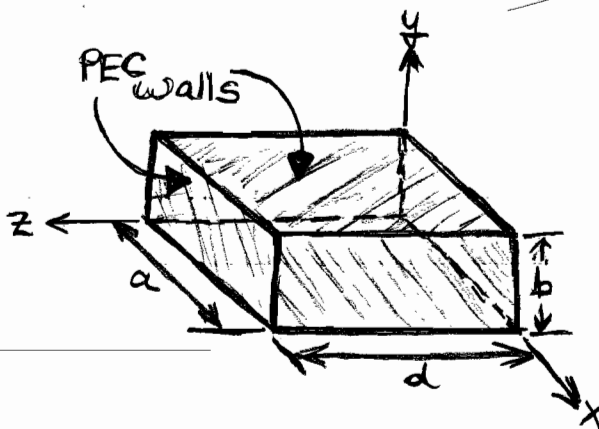


Resonant devices with extremely high Q can be realized by placing metal end caps onto a rectangular waveguide.



Energy is coupled into the cavity through a small hole (or holes) on a side wall, or by a small wire or loop extended through a side wall. We'll assume the excitation occurs on a $z = \text{constant}$ side wall.

A wave excited at $z = 0$, for example, may prop. as a waveguide mode in $+z$, reflect off the wall at $z = d$, then prop. back to the $z = 0$ wall.

At certain frequencies, these incident and reflected waves will add together in phase, strengthening the oscillations. Other than at these special frequencies, the waves interfere destructively ^{inside the cavity} & so no energy can be coupled into the cavity.

We can determine these special frequencies by applying b.c.'s at $z = 0$ & $z = d$.

... We must be a bit careful here to apply the b.c.'s to the total \vec{E}_{tan} at these surfaces.

... TE^z cavity modes. With propagation in the $\pm z$ directions, then

... For +z prop:
$$H_z^+ = A_{mn} \cos_x^a \cos_y^b e^{-j\beta_z z} \quad (1)$$

...
$$E_x^+ = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} A_{mn} \cos_x^a \sin_y^b e^{-j\beta_z z} \quad (2)$$

...
$$E_y^+ = \frac{-j\omega\mu\beta_{xm}}{\beta_{cmn}^2} A_{mn} \sin_x^a \cos_y^b e^{-j\beta_z z} \quad (3)$$

... For prop. in the $-z$ direction, $\beta_z \rightarrow -\beta_z$ in (1)-(3) gives

...
$$H_z^- = B_{mn} \cos_x^a \cos_y^b e^{+j\beta_z z} \quad (4)$$

...
$$E_x^- = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} B_{mn} \cos_x^a \sin_y^b e^{+j\beta_z z} \quad (5)$$

...
$$E_y^- = \frac{-j\omega\mu\beta_{xm}}{\beta_{cmn}^2} B_{mn} \sin_x^a \cos_y^b e^{+j\beta_z z} \quad (6)$$

... Using (2) & (5), the total $E_x^t = E_x^+ + E_x^-$ ^{for mode mn} is

...
$$E_x^t = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} \cos_x^a \sin_y^b (A_{mn} e^{-j\beta_z z} + B_{mn} e^{+j\beta_z z}) \quad (7)$$

... E_x^t already satisfies b.c.'s at $x=0, a$ and $y=0, b$:

... That's how we originally determined $\beta_{xm} = \frac{m\pi}{a}$ & $\beta_{yn} = \frac{n\pi}{b}$.

... Now need to apply boundary conditions at $z=0$ & d

... for $\vec{E}_{tan} = 0$.

at $z=0$
 eq (7) For $E_x^t = 0 = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} \cos^a \sin^b (A_{mn} + B_{mn})$

$\Rightarrow B_{mn} = -A_{mn}$ (8)

Sub. (8) into (7):

$E_x^t = \frac{j\omega\mu\beta_{yn}}{\beta_{cmn}^2} A_{mn} (\cos^a \sin^b (e^{-j\beta_z z} - e^{+j\beta_z z}))$
 $= -j2 \sin(\beta_z z)$

or
 $E_x^t = \frac{2j\omega\mu\beta_{yn}}{\beta_{cmn}^2} A_{mn} \cos^a \sin^b \sin(\beta_z z)$ (9)

At $z=d$, $E_x^t = 0 = \frac{2j\omega\mu\beta_{yn}}{\beta_{cmn}^2} A_{mn} \cos^a \sin^b \sin(\beta_z d)$

$\Rightarrow \beta_z = \beta_{zp} = \frac{p\pi}{d}$ $p=1, 2, 3, \dots$ (10)

It can be shown that this same condition (10) will also enforce the b.c. $E_y = 0$ at $z=0, d$ since using (3) & (6)

$E_y^t = E_y^+ + E_y^- = \frac{-2j\omega\mu\beta_{xm}}{\beta_{cmn}^2} A_{mn} \sin^a \cos^b \sin(\beta_z z)$ (11)

With $\beta_z = \beta_{zp}$ in (10), E_y^t in (11) vanishes at $z=0, d$.

Notice that in (10), $p \neq 0$. If $p=0$ in (9) or (11), then $E_x^t = E_y^t = 0 \forall x, y, z$ inside the cavity. Because $E_z = 0$ for the TE modes, then $\vec{E} = 0$ if $p=0$. With $\vec{E} = 0$, no resonance is possible. (Can also show that $H_z = 0$ for

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$$

$$\beta_z = \sqrt{\beta^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \text{assume any value.}$$

A/7

$\rho=0 \Rightarrow$ all TE_z^2 fields vanish.)

Substituting (10) into the dispersion relation

$$\beta_{xm}^2 + \beta_{yn}^2 + \beta_{zp}^2 = \beta^2 = \omega^2 \mu \epsilon$$

then

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 = \omega^2 \mu \epsilon \quad (12)$$

The only degree of freedom left is the frequency of operation.

From (12), the resonant frequencies of the cavity are thus

$$f_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2} \quad (13)$$

for TE_{mnp}^2 modes where $\checkmark m, n = 0, 1, 2, \dots$ ($m=n \neq 0$), $\checkmark p = 1, 2, 3, \dots$

● TM^z cavity modes. Referencing Pozar, 3rd edition, ch. 3, eqns. (3.100) and (3.101), for prop. in the +z direction:

$$E_z^+ = A_{mn} \sin_x^a \sin_y^b e^{-j\beta_{zmn} z} \quad (14)$$

and

$$E_x^+ = \frac{-j\beta_z}{\beta_c^2} \frac{\partial E_z}{\partial x} = \frac{-j\beta_{zmn}}{\beta_{cmn}^2} A_{mn} \beta_{xm} \cos_x^a \sin_y^b e^{-j\beta_{zmn} z} \quad (15)$$

$$E_y^+ = \frac{-j\beta_z}{\beta_c^2} \frac{\partial E_z}{\partial y} = \frac{-j\beta_{zmn}}{\beta_{cmn}^2} A_{mn} \beta_{yn} \sin_x^a \cos_y^b e^{-j\beta_{zmn} z} \quad (16)$$

For propagation in the -z direction ($\beta_{zmn} \rightarrow -\beta_{zmn}$) then

$$E_z^- = B_{mn} \sin_x^a \sin_y^b e^{+j\beta_z z} \quad (17)$$

and

$$E_x^- = \frac{+j\beta_z \beta_{xm}}{\beta_{cmn}^2} B_{mn} \cos_x^a \sin_y^b e^{+j\beta_{zmn} z} \quad (18)$$

$$E_y^- = \frac{+j\beta_z \beta_{ym}}{\beta_{cmn}^2} B_{mn} \sin_x^a \cos_y^b e^{+j\beta_{zmn} z} \quad (19)$$

Comparing (15), (16), (18), & (19) with the transverse \vec{E} for TE^z modes in (2), (3), (5), & (6), we see that the transverse \vec{E} for TE^z & TM^z modes have the same spatial dependencies. Consequently, applying boundary conditions at $z=0$ & d , will arrive at the same resonant frequency expression as in (13).

It turns out, though, that the index $p=0$ is allowed for TM^z modes. How can this be since $E_x = E_y = 0 \forall x, y, z$ when $p=0$? For TM^z modes, $E_z \neq 0$. From (14) and (17) with $B_{mn} = A_{mn}$:

$$\begin{aligned} E_z^t &= E_z^+ + E_z^- = A_{mn} \sin_x^a \sin_y^b (e^{-j\beta_{zp} z} + e^{+j\beta_{zp} z}) \\ &= 2 A_{mn} \sin_x^a \sin_y^b \cos(\beta_{zp} z) \end{aligned} \quad (20)$$

$$\text{For } p=0, \quad E_z^t = 2 A_{mn} \sin_x^a \sin_y^b \neq 0.$$

So, for TM^z modes with $p=0$, E_z^t is constant in z but varies in x & y .

What about \vec{H} ? Can show that $H_x, H_y \propto \cos(\frac{p\pi}{d} z)$

... such that $H_x, H_y \neq 0$. Consequently, $E_z, H_x,$ and H_y are non-zero so that TM_{mnp}^z modes are possible.

... Consequently, for TM_{mnp}^z modes, the resonant frequencies are

$$f_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2} \quad (21)$$

... where $m, n = 1, 2, 3, \dots$;
 $p = 0, 1, 2, \dots$

Dominant Mode in a Rectangular Cavity Resonator

... For the case when the cavity dimensions are related by $b < a < d$, as sketched on p. 1, (13) and (21) to determine that the cavity mode with the lowest resonant frequency is TE_{101} . This is the dominant or fundamental cavity mode.

... The TE_{101} mode corresponds to a TE_{10} waveguide mode in a $\frac{\lambda_g}{2}$ -long cavity.

... Which mode resonates next as the frequency increases depends on the specific ratios of $\frac{b}{a}$ and $\frac{d}{a}$.

For example, in the case of a WR-90 waveguide that is 2-cm long in z :

<u>Resonant frequency (GHz)</u>	<u>mode</u>
9.959	TE ₁₀₁ ✓
15.105	TE ₂₀₁ ✓
16.146	TM ₁₁₀
16.362	TE ₁₀₂ ✓
16.549	TE ₀₁₁ ✓
17.800	TE ₁₁₁ , TM ₁₁₁ ✓
19.740	TM ₂₁₀
19.917	TE ₂₀₂ ✓

These are the resonant frequencies through 20 GHz. Notice:

- As the frequency increases, the "density" of resonant modes increases.
- At 17.800 GHz, both TE₁₁₁ & TM₁₁₁ can resonate. Two modes that have the same resonant frequency but different field patterns are called degenerate modes.

```

a = 0.9 * 2.54 / 100. ;
b = 0.4 * 2.54 / 100. ;
d = 0.02 ;
c0 = 2.998 * 108 ;
fres[m_, n_, p_] := c0 / (2. * Pi) * Sqrt[(m * Pi / a)2 + (n * Pi / b)2 + (p * Pi / d)2]

```

TEz modes (where m and n can not simultaneously equal zero).

```

Table[{m, n, p, fres[m, n, p] / 109}, {m, 0, 4}, {n, 0, 4}, {p, 1, 4}] ;
Prepend[%, {"m", "n", "p", "fres (GHz)"}] ;
TableForm[%]

```

Out[32]//TableForm=

				<i>TE_{mnp}</i>			
m	n	p	fres (GHz)	m	n	p	fres (GHz)
0	0	1	7.495	0	1	1	16.5485
0	0	2	14.99	0	1	2	21.0328
0	0	3	22.485	0	1	3	26.8934
0	0	4	29.98	0	1	4	33.4138
1	0	1	9.95858	1	1	1	17.8003
1	0	2	16.3615	1	1	2	22.0313
1	0	3	23.4216	1	1	3	27.6813
1	0	4	30.6887	1	1	4	34.0511
2	0	1	15.1052	2	1	1	21.1151
2	0	2	19.9172	2	1	2	24.7865
2	0	3	26.0301	2	1	3	29.9207
2	0	4	32.723	2	1	4	35.8953
3	0	1	21.0513	3	1	1	25.7068
3	0	2	24.7323	3	1	2	28.7987
3	0	3	29.8757	3	1	3	33.3202
3	0	4	35.8578	3	1	4	38.7745
4	0	1	27.2791	4	1	1	31.0133
4	0	2	30.2105	4	1	2	33.6207
4	0	3	34.5478	4	1	3	37.5663
4	0	4	39.8343	4	1	4	42.4788
0	4	1	44.8919	0	2	1	30.4449
0	4	2	60.8897	0	2	2	33.0971
0	4	3	63.154	0	2	3	37.0984
0	4	4	66.1941	0	2	4	42.0656
1	4	1	59.8501	1	2	1	31.143
1	4	2	61.2418	1	2	2	33.7404
1	4	3	63.4936	1	2	3	37.6734
1	4	4	66.5181	1	2	4	42.5736
2	4	1	60.9182	2	2	1	33.1494
2	4	2	62.286	2	2	2	35.6007
2	4	3	64.5014	2	2	3	39.3482
2	4	4	67.4808	2	2	4	44.0625
3	4	1	62.6579	3	2	1	36.2474
3	4	2	63.9886	3	2	2	38.5019
3	4	3	66.1469	3	2	3	41.9914
3	4	4	69.0554	3	2	4	46.4381
4	4	1	65.0154	4	2	1	40.1853
4	4	2	66.2988	4	2	2	42.2302
4	4	3	68.3843	4	2	3	45.4341
4	4	4	71.2013	4	2	4	49.573

TMz modes


```
In[33]:= Table[{m, n, p, fres[m, n, p]/10^9}, {m, 1, 4}, {n, 1, 4}, {p, 0, 4}] ;
Prepend[%, {"m", "n", "p", "fres (GHz)"}] ;
TableForm[%]
```

T_{mnp}

```
Out[35]//TableForm=
```

m	n	p	fres (GHz)
1	1	0	16.1455
1	1	1	17.8003
1	1	2	22.0313
1	1	3	27.6813
1	1	4	34.0511
2	1	0	19.7401
2	1	1	21.1151
2	1	2	24.7865
2	1	3	29.9207
2	1	4	35.8953
3	1	0	24.5899
3	1	1	25.7068
3	1	2	28.7987
3	1	3	33.3202
3	1	4	38.7745
4	1	0	30.094
4	1	1	31.0133
4	1	2	33.6207
4	1	3	37.5663
4	1	4	42.4788
1	2	0	30.2277
1	2	1	31.143
1	2	2	33.7404
1	2	3	37.6734
1	2	4	42.5736
2	2	0	32.291
2	2	1	33.1494
2	2	2	35.6007
2	2	3	39.3482
2	2	4	44.0625
3	2	0	35.4641
3	2	1	36.2474
3	2	2	38.5019
3	2	3	41.9914
3	2	4	46.4381
4	2	0	39.4802
4	2	1	40.1853
4	2	2	42.2302
4	2	3	45.4341
4	2	4	49.573
1	3	0	44.7449
1	3	1	45.3683
1	3	2	47.189
1	3	3	50.0768
1	3	4	53.8601
2	3	0	46.1638
2	3	1	46.7683
2	3	2	48.5366
2	3	3	51.3486
2	3	4	55.0445
3	3	0	48.4365
3	3	1	49.0129
3	3	2	50.703
3	3	3	53.401
3	3	4	56.964
4	3	0	51.4498
4	3	1	51.9928
4	3	2	53.589
4	3	3	56.1485
4	3	4	59.5473
1	4	0	59.3789
1	4	1	59.8501
1	4	2	61.2418
1	4	3	63.4936
1	4	4	66.5181
2	4	0	60.4554
2	4	1	60.9182
2	4	2	62.286
2	4	3	64.5014
2	4	4	67.4808
3	4	0	62.2081
3	4	1	62.6579
3	4	2	63.9886
3	4	3	66.1469
3	4	4	69.0554
4	4	0	64.582
4	4	1	65.0154
4	4	2	66.2988
4	4	3	68.3843
4	4	4	71.2013