Similar to TEM waves, it is possible to identify transmission line equivalent representations for TE and TM modes in waveguides. While this is possible, there are important differences in the physical interpretation of equivalent voltage, current, and impedance.

In particular, for waveguides:

- Voltage and current are defined for a specific waveguide mode \((m, n)\), where the equivalent voltage is proportional to the transverse electric field \(E_t\), and the equivalent current is proportional to the transverse magnetic field \(H_t\).
- The ratio of \(V\) to \(I\) for a \(\pm\) propagating mode equals the wave impedance in the waveguide. Are there such a scalar quantity? Need to see.
- The product \(\pm\frac{1}{2}V \cdot I^*\) should yield power propagating in \(\pm\) directions, respectively, in a waveguide for a specific propagating mode.

Equivalent admittance

As defined above, the ratio of the transverse \(E_t\) and \(H_t\) should equal the wave admittance. We'll examine this for \(TE\) and \(TM\) modes in hollow metallic waveguides.
For TE modes: We've shown previously for $e^{-j\beta z}$ propagation that

\[ E_x = -j \frac{\omega \mu}{c^2} \frac{\partial H_y}{\partial t} \quad (1) \]

\[ E_y = \frac{j \omega \mu}{c^2} \frac{\partial H_x}{\partial x} \quad (3) \]

\[ H_y = \frac{j \beta_z}{c^2} \frac{\partial H_x}{\partial t} \quad (2) \]

\[ H_x = -\frac{j \beta_z}{c^2} \frac{\partial H_y}{\partial x} \quad (4) \]

... Dividing (1) by (2) for $\frac{\partial H_x}{\partial t} \neq 0$:

\[ \frac{E_x}{H_y} = \frac{\omega \mu}{c^2} \cdot \frac{\beta_z}{c^2} = \frac{\omega \mu}{\beta_z} \equiv Z_{TE} \quad (5) \]

... and dividing (3) by (4) for $\frac{\partial H_y}{\partial x} \neq 0$:

\[ \frac{E_y}{H_x} = \frac{\omega \mu}{c^2} \cdot \left( -\frac{\beta_z}{c^2} \right) = -\frac{\omega \mu}{\beta_z} \equiv -Z_{TE} \quad (6) \]

... More compactly, we can express (5) and (6) in this one equation

\[ Z_{TE} = \frac{1}{\varepsilon_{TE}} \varepsilon \times \bar{E}_{TE} \quad (7) \]

\[ = \frac{1}{Z_{TE}} \varepsilon \times \left( \frac{\lambda}{Z_{TE}} \bar{E}_{x,TE} + \frac{\lambda}{Z_{TE}} \bar{E}_{y,TE} \right) \]

This mode impedance $Z_{TE} = \frac{\omega \mu}{\beta_z}$ is a function of frequency, shape, as well as the mode indices $m, n$.

For TM modes: We've shown previously for $e^{-j\beta z}$ propagation that
\[
E_x = -\frac{i\beta_x}{\beta_z} \frac{\partial E_z}{\partial x} \quad (8), \quad E_y = -\frac{i\beta_x}{\beta_z} \frac{\partial E_z}{\partial y} \quad (10)
\]

\[
H_x = \frac{-i\omega e}{\beta_z} \frac{\partial E_x}{\partial x} \quad (9), \quad H_y = \frac{i\omega e}{\beta_z} \frac{\partial E_x}{\partial y} \quad (11)
\]

...Dividing (8) by (9) for \( \frac{\partial E_z}{\partial x} \neq 0 \):

\[
\frac{E_x}{H_x} = -\frac{\beta_x}{\beta_z} \cdot \frac{\beta_z}{-j\omega e} = \frac{\beta_x}{\omega e} \equiv Z_{TM} \quad (12)
\]

...while dividing (10) by (11) for \( \frac{\partial E_z}{\partial y} \neq 0 \):

\[
\frac{E_y}{H_x} = \frac{\beta_x}{\beta_z} \cdot \frac{\beta_z}{j\omega e} = -\frac{\beta_x}{\omega e} \equiv -Z_{TM} \quad (13)
\]

Eqs. (12) and (13) can be expressed more compactly as

\[
T_{TM} = \frac{1}{Z_{TM}} \hat{x} \cdot \hat{E}_{TM} \quad (14)
\]

\[
\left[ = \frac{1}{Z_{TM}} \hat{x} \left( \hat{x} E_{x,TM} + \hat{y} E_{y,TM} \right) \right] \hat{x}
\]

As for \( TE \), this \( TM \) mode impedance is also a function of frequency and the mode indices \( m \) and \( n \). Note that (9) and (11) have identical forms; only the modal impedances differ.
Equivalent Voltages & Currents

In the case of both +z and -z prop. waves in a waveguide, we can express the transverse $\mathbf{E} \times \mathbf{H}$ for a specific mode $(m,n)$ as:

$$
\mathbf{E}_\pm(x,y,z) = \mathbf{E}(x,y) \left( A e^{-j\beta_z z} + A^* e^{+j\beta_z z} \right)
$$

leads to $V(z)$

(4.6a), (15)

and

$$
\mathbf{H}_\pm(x,y,z) = \mathbf{H}(x,y) \left( A^* e^{-j\beta_z z} - A e^{+j\beta_z z} \right)
$$

leads to $I(z)$

(4.6b), (12)

where we've just found what for a particular mode $(m,n)$

$$
\mathbf{H}(x,y) = \frac{1}{\mathbf{r}_w} \mathbf{r}_w \times \mathbf{E}(x,y)
$$

(4.7), (17)

and $\mathbf{r}_w = \mathbf{r}_{TE}$ or $\mathbf{r}_{TM}$ as appropriate.

From (15) (16) we can quickly identify scalar equations that are directly proportional to transverse $\mathbf{E} \times \mathbf{H}$ and depend on $z$ only. In particular, we'll derive from (15)

$$
\mathbf{E}_\pm(x,y,z) = \mathbf{E}(x,y) \cdot \frac{V(z)}{C_z}
$$

(18)

while from (16)

$$
\mathbf{H}_\pm(x,y,z) = \mathbf{H}(x,y) \cdot \frac{I(z)}{C_z}
$$

(19)

So,

$$
V(z) = V^+ e^{-j\beta_z z} + V^- e^{+j\beta_z z}
$$

(4.8a), (20)

and

$$
I(z) = I^+ e^{-j\beta_z z} - I^- e^{+j\beta_z z}
$$

(4.8b), (21)

Comparing (20), (18) & (15) we find that

$$
\frac{V^+}{C_z} = \mathbf{A}^+ \quad \frac{V^-}{C_z} = \mathbf{A}^-
$$
\[ C_1 = \frac{V^+}{A^+} = \frac{V^-}{A^-} \quad (22) \]

Similarly, comparing (21), (19), and (14)

\[ C_2 = \frac{I^+}{A^+} = \frac{I^-}{A^-} \quad (23) \]

Further, the ratio \( \frac{V^+}{I^+} \) is associated with the wave impedance. From (22) and (23),

\[ \frac{C_1}{C_2} = \frac{\frac{V^+}{A^+}}{\frac{I^+}{A^+}} = \frac{\frac{V^+}{I^+}}{\frac{I^+}{A^+}} = \frac{V^+}{I^+} = Z_w \quad (24) \]

Similarly, from (22) and (23):

\[ \frac{C_1}{C_2} = \frac{V^-}{I^-} = Z_w \quad (25) \]

Hence, the equivalent TL model agrees for the shallow metallic waveguide as from (20), (20) using (24) and (25).

\[ V(z) = V^+ e^{-j k_w z} + V^- e^{+j k_w z} \quad (26) \]

\[ I(z) = \frac{V^+}{Z_w} e^{-j k_w z} - \frac{V^-}{Z_w} e^{+j k_w z} \quad (27) \]

These TL equations are very similar to those that model TEM modes. Primary differences are:

- The TL is intrinsically dispersive because \( k_w \)
  - a fact of free space

- The TL impedance is intrinsically frequency dependent

- This TL models a single mode \((m, n, p)\)
  - propagating

- The equivalent \( V(z) \) and \( I(z) \) are proportional to the
The utility of the TL model for hollow metallic waveguides is that we can use it to simplify the analysis of certain waveguide problems. An example of such a problem is the joint of two homogeneously filled waveguides, as in the next example.

\[ \text{(Text example 4.2)} \]

**Example:** A WR-90 waveguide is half filled by Ferrolyte, as shown below. Determine the electric field reflection coefficient \( V \) at \( z = 0 \) using an equivalent TL model for a freq. of 8 GHz. Plot over X band.

\[ Z_{eq} > Z_{0,0} \]
For WR-90 waveguide, \( a = 2.286 \text{ cm (0.9")} \) and \( b = 1.014 \text{ cm (0.4")} \).

For the \( TE_{mn} \) modes,

\[
\frac{f_{mn}}{c_0} = \frac{1}{2\pi a} \frac{\sqrt{n^2 \epsilon_r - \left(\frac{m \pi}{a}\right)^2}}{\epsilon_r} = \frac{MC_0}{2a \sqrt{\epsilon_r}}
\]

The two lowest-order modes in the two waveguides are:

- **Air (\( \epsilon_r = 1 \))**
  - \( m = 1 \): \( f_{c_{10}} = 6.557 \text{ GHz} \)
  - \( m = 2 \): \( f_{c_{20}} = 13.11 \text{ GHz} \)

- **Resolda (\( \epsilon_r = 2.54 \))**
  - \( f_{c_{10}} = 4.11 \text{ GHz} \)
  - \( f_{c_{20}} = 8.23 \text{ GHz} \)

At 8 GHz, only \( TE_{10} \) mode propagates in both waveguides.

For \( TE^z \) modes, the wave speed \( c_z \) is given in (5) to be

\[
Z_{TE} = \frac{\omega c_z}{\beta_z}
\]

For the \( \epsilon_z \) guided TEs,

\[
Z_{0a} = \frac{\omega c_0}{\beta_{c_{10}}} = \frac{\omega c_0}{\sqrt{\beta_0^2 - \left(\frac{m \pi}{a}\right)^2}} = 657.6 \Omega
\]

and

\[
Z_{0b} = \frac{\omega c_0}{\sqrt{\beta_0^2 \epsilon_r - \left(\frac{m \pi}{a}\right)^2}} = 275.6 \Omega
\]

Smaller than because "farther from cut-off".

The electric field ref. coeff. can be computed from the \( \epsilon_z \)-guided TE model in the traditional fashion as

\[
\Gamma = \frac{Z_{0b} - Z_{0a}}{Z_{0b} + Z_{0a}} = -0.41 \text{ at } 8 \text{ GHz}
\]
\((\text{Why is the equivalent to the electric field ref coeff. in the wire?})\)

... Plot of \(P\) vs. \(f\) is shown on next page over the frequency range where single mode prop. in both guides. Very large ref. @ \(\nu\text{c, } \nu\text{g}\) because guide at near cutoff.

As \(f\) increases, thinking of boundary range, \(P\) should approach that of a UPW normally incident on a half space of substrate.

\[
\Gamma_{\text{TEM}} = \frac{n_r-1}{n_r+1} = \frac{\sqrt{\varepsilon_r} - 1}{\sqrt{\varepsilon_r} + 1} = -0.229.
\]
In[75] := ClearAll[e0, μ0, GHZ, a, b, βz, m, n, f, er, ZOTE, Γ, er1, er2, freq]

In[76] := a = 0.9*2.54/100.;
b = 0.4*2.54/100.;
e0 = 8.854*10^-12 ;
μ0 = 4.*Pi*10^-7 ;
GHZ = 10^3 ;

In[81] := βz[m_, n_, f_, er_] := Sqrt[(2.*Pi*f)^2*μ0*e0*er - (m*Pi/a)^2 - (n*Pi/b)^2]
ZOTE[m_, n_, f_, er_] := 2.*Pi*f*μ0/βz[m, n, f, er]
Γ[m_, n_, f_, er1_, er2_] :=
(ZOTE[m, n, f, er2] - ZOTE[m, n, f, er1]) / (ZOTE[m, n, f, er2] + ZOTE[m, n, f, er1])

In[84] := er1 = 1.;
er2 = 2.54 ;
freq = 8.*GHZ ;
βz[1, 0, freq, er1]
βz[1, 0, freq, er2]
ZOTE[1, 0, freq, er1]
ZOTE[1, 0, freq, er2]
Γ[1, 0, freq, er1, er2]

Out[87] = 96.0495 = βz_2, a
Out[88] = 229.167 = βz_2, b
Out[89] = 657.634 = ZOTE_0, a
Out[90] = 275.63 = ZOTE_0, b
Out[91] = -0.40932 = Γ

In[100] :=
Plot[Γ[1, 0, f*GHZ, er1, er2], {f, 6.558, 8.229},
AxesLabel -> {f [GHZ], "Γ"}, PlotStyle -> {RGBColor[1, 0, 0]}, PlotRange -> All]