...The dispersion relation for a rectangular waveguide is very similar for TE\(_m^n\) and TM\(_m^n\) modes. We found in the last lecture, for wave prop x + z, that

\[ \beta_x \pm \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2 \mu \varepsilon \]

(1)

where

\[ \beta_x = \frac{m \pi}{a}, \quad m = 0, 1, 2, \ldots \]

(2)

\[ \beta_y = \frac{n \pi}{b}, \quad n = 1, 2, \ldots \]

(3)

...and for TE\(_m^n\) modes \( m = n \neq 0 \) while for TM\(_m^n\) modes \( m \neq 0 \) \& \( n \neq 0 \).

...The form of the wave propagation x + z is \( e^{-j \beta_z z} \)

...and the longitudinal wave number is from (1)

\[ \beta_z = \pm \sqrt{\beta^2 - \beta_x^2 - \beta_y^2} \]

(4)

...There are an infinite # of these modes \( m \) \& \( n \) that are generally "excited" when a source is applied to the waveguide. Each of these modes is identified by it's indices \( m \) \& \( n \) as TE\(_m^n\) \& TM\(_m^n\).

...How "much" of each mode contributes to the total field depends on the specific excitation.

... Launcher Amplifier
We will study the general solutions to the waveguide (the eigenvalues) and not concern ourselves by the general excitation, very difficult analysis.

1. See R.E. Collin, "Field Theory of Guided Waves."

These modes in the waveguide have a very unusual characteristic we can see from (4).

We'll define

\[ \beta_{mn}^2 = \beta_m^2 + \beta_n^2 \]  \hspace{1cm} (5)

So that (4) reads

\[ \beta_{2mn} = \pm \sqrt{\beta_m^2 - \beta_{mn}^2} \]  \hspace{1cm} (6)

Notice in (6) that when \( \beta_m^2 < \beta_{mn}^2 \) the wave is purely attenuated while when \( \beta_m^2 > \beta_{mn}^2 \) the wave is purely propagating. \( \beta_m^2 \) is a function of frequency (\( \varepsilon \mu \omega^2 \)) while \( \beta_{mn}^2 \) is dependent on dimensions a, b as well as indices m,n.

When \( \beta_m^2 = \beta_{mn}^2 \) the wave is neither propagating nor attenuating. It is called cut off. For this reason, \( \beta_{mn} \) is called the cutoff wavenumber.

Hence, from (5)

\[ \beta_{mn} = \sqrt{\left( \frac{\omega \mu}{a} \right)^2 + \left( \frac{\omega \varepsilon}{b} \right)^2} \]  \hspace{1cm} (7)

Also, from (6)

\[ \beta_{mn} = \beta_i \left|_{\omega_m \varepsilon M} \right. = 2\pi f_{c,0} \left( \frac{\omega_m \varepsilon M}{\beta_{mn}} \right) \]  \hspace{1cm} (8)

where \( f_{c,mn} \) is the cutoff frequency of mode m,n.
Using (8) in (7),
\[ f_{mn} = \frac{1}{2\pi\sqrt{M}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \]

...let's take an X-band rectangular waveguide as an example. This is called a WR-90 waveguide.

From Table B.4, inside dimensions \( a = 0.90" \) (22.86 cm) & \( b = 0.40" \) (1.016 cm). Assuming air-filled.

The first 10 modes are:

1. \( TE_{10} \) \( \quad f_{10} = 6.562 \) GHz
2. \( TE_{20} \) \( \quad f_{20} = 13.124 \) GHz
3. \( TE_{01} \) \( \quad f_{20} = 14.764 \) GHz
4. \( TE_{11} \) \( \quad f_{11} = 16.56 \) GHz \( \Rightarrow \) degenerate modes
5. \( TM_{11} \)
6. \( TE_{30} \) \( \quad f_{30} = 19.685 \) GHz
7. \( TE_{21} \) \( \quad f_{21} = 19.754 \) GHz \( \Rightarrow \) degenerate
8. \( TM_{21} \)
9. \( TE_{31} \) \( \quad f_{31} = 24.607 \) GHz \( \Rightarrow \) degenerate
10. \( TM_{31} \)

The mode with the smallest \( f_c \) is called the dominant mode. For this waveguide having \( a > b \),
the dominant mode is \( TE_{10} \).

Let's plot \( \beta_{210} \).
For $f < f_{c10}$, no modes prop. They are purely attenuated. (Fields near source are not zero, however, if this mode.

They are reactive fields yielding realistic output impedance.)

For $f_{c10} < f < f_{c20}$ only one mode will propagate.

All others purely attenuated. Most often will operate W9D3 in single mode. For this particular W9D3,

Single mode operation is possible from 0.862 GHz to 13.126 GHz.

This is assuming no loss. Metal losses tend to "smear out" the response such as...
There is another characteristic of wave propagation in hollow metallic waveguides that is very apparent from these plots of $\beta_{mn}$. These waves are dispersive, even when there are no losses. (Metalllic losses would add further dispersion.)

That is, $\omega \in \beta_{mn}$ in propagation, the phase velocity is dependent on frequency. By definition, the phase velocity $V_p$ is given as

$$V_p = \frac{\omega}{k}$$

For propagation in $+z$, the phase velocity in that medium is

$$V_{p2} = \frac{\omega}{\beta_{2mn}} = \frac{\omega}{\sqrt{\beta^2 - \beta_{2mn}^2}} \quad \beta > \beta_{2mn} \quad (9)$$
Until plane waves in lossless media, $\nu_p$ is a function of frequency.

Similarly, the group velocity is also a function of $f$. This velocity is defined as the inverse slope of the dispersion curve ($\beta = \omega$):

$$\nu_g = \frac{d\omega}{d\beta}$$

Group velocity is speed of information transmission.

It is interesting to plot the phase and group velocities (Fig 8-8):

![Diagram showing phase and group velocities](image-url)