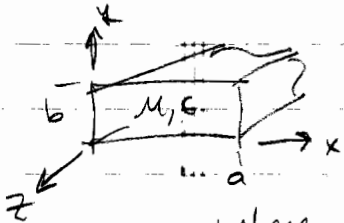


Lecture 26 - General Behavior
of Rectangular Waveguides

The dispersion relation for a rectangular waveguide is very similar for TE^z & TM^z modes. We found in the last lecture, for wave prop in $+z$, that



$$\beta_{xm}^2 + \beta_{yn}^2 + \beta_{zmn}^2 = \beta^2 = \omega^2 \mu \epsilon \quad (1)$$

where

$$\beta_{xm} = \frac{m\pi}{a} \quad m = 0, 1, 2, \dots \quad \begin{cases} m=n \neq 0 & TE^z \\ m \neq 0 & TM^z \end{cases} \quad (2)$$

$$\beta_{yn} = \frac{n\pi}{b} \quad n = 0, 1, 2, \dots \quad (3)$$

and for TE^z modes $m=n \neq 0$ while for TM^z modes $m \neq 0$ & $n \neq 0$.

The form of the wave propagation in $+z$ is $e^{-j\beta_{zmn}z}$ and the longitudinal wave number is from (1)

$$\beta_{zmn} = \pm \sqrt{\beta^2 - \beta_{xm}^2 - \beta_{yn}^2} \quad (4)$$

There are an infinite # of these modes mn that are generally "excited" when a source is applied to the waveguide. Each of these modes is identified by it's indices m & n as TE_{mn}^z & TM_{mn}^z . How "much" of each mode contributes to the total field depends on the specific excitation.



We will study the general solutions to the waveguide (the eigenspectrum) & not concern ourselves w/ the general excitation. Very difficult analysis. (See R.E. Collin, "Field Theory of Guided Waves".)

These modes in the waveguide have a very unusual characteristic we can see from (4).

We'll define

$$\beta_{c_{mn}}^2 = \beta_{x_{mn}}^2 + \beta_{y_{mn}}^2 \quad (5)$$

So that (4) reads

$$\beta_{z_{mn}} = \pm \sqrt{\beta^2 - \beta_{c_{mn}}^2} \quad (6)$$

Notice in (6) that when $\beta^2 < \beta_{c_{mn}}^2$ the wave is purely attenuated while when $\beta^2 > \beta_{c_{mn}}^2$ the wave is purely propagating. β^2 is a function of frequency ($\neq \mu \neq \epsilon$) while $\beta_{c_{mn}}^2$ is dependent on dimensions a, b as well as indices m, n .

When $\beta^2 = \beta_{c_{mn}}^2$ the wave is neither propagating nor attenuating. It is called cut off. For this reason, $\beta_{c_{mn}}$ is called the cutoff wavenumber.

$$\text{Hence, from (5)} \quad \beta_{c_{mn}} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad (7)$$

$$\text{Also, from (6)} \quad \beta_{c_{mn}} = \beta \Big|_{\beta_{z_{mn}}=0} = \omega_{c_{mn}} \sqrt{\mu\epsilon} = 2\pi f_{c_{mn}} \sqrt{\mu\epsilon} \quad (8)$$

where $f_{c_{mn}}$ is the cutoff frequency of mode mn .

Using (8) in (7), $f_{c_{mn}} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ (8-16), (9)

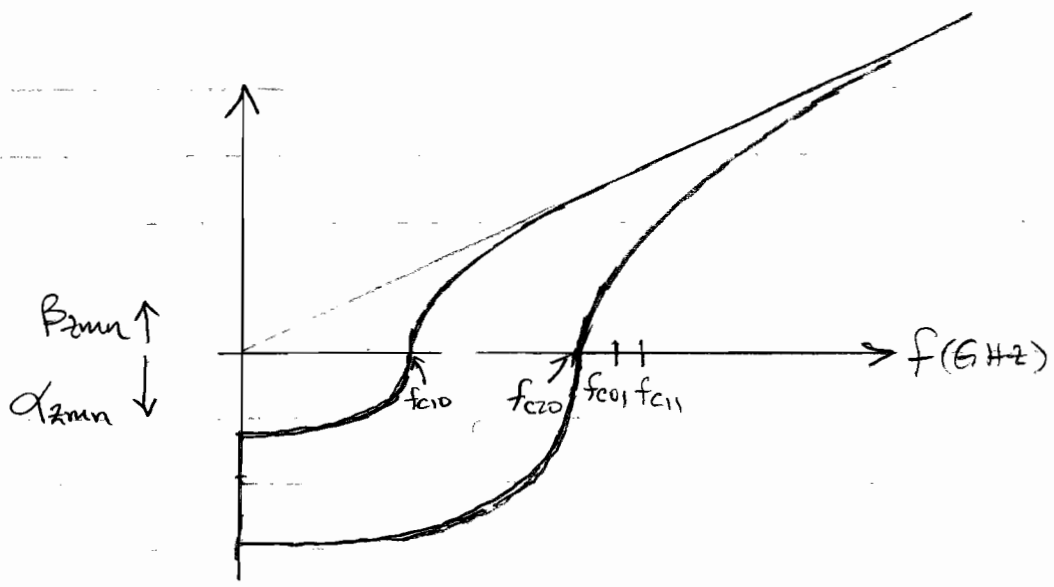
Let's take an X-band rectangular waveguide as an example. This is called a WR-90 waveguide.

From Table 8-4, Inside dimensions $a = 0.90''$ (2.286 cm) & $b = 0.40''$ (1.016 cm). ^{Assuming air-filled,} The first 10 modes with the lowest cutoff frequencies are (Example 8-2):

1.	TE ₁₀	$f_{c10} = 6.562$ GHz	
2.	TE ₂₀	$f_{c20} = 13.124$ GHz	
3.	TE ₀₁	$f_{c01} = 14.764$ GHz	
4.	TE ₁₁	$f_{c11} = 16.56$ GHz	} degenerate modes
5.	TM ₁₁	"	
6.	TE ₃₀	$f_{c30} = 19.685$ GHz	
7.	TE ₂₁	$f_{c21} = 19.754$ GHz	} degenerate
8.	TM ₂₁	"	
9.	TE ₃₁	$f_{c31} = 24.607$ GHz	} degenerate
10.	TM ₃₁	"	

The mode with the smallest f_c is called the dominant mode. For this waveguide having $a > b$, the dominant mode is TE₁₀.

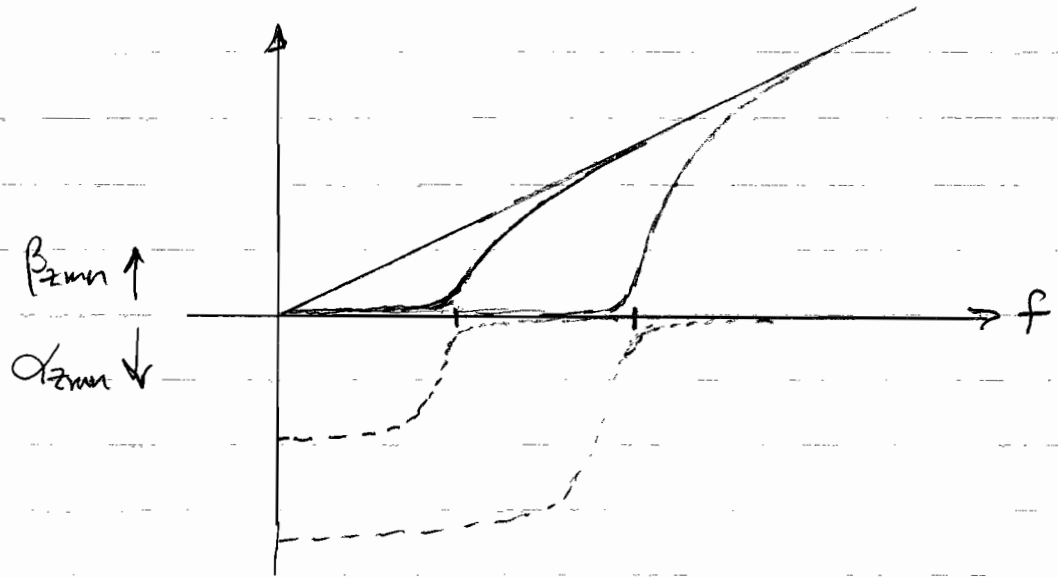
Let's plot β_{z10} .



... For $f < f_{c10}$, no modes prop. They are purely attenuated. (Fields near source are not zero, however, they are reactive ^{type of this mode} fields yielding reactive input impedance)

... For $f_{c10} < f < f_{c20}$ only one mode will propagate. All others purely attenuated. Most often will operate in single mode. For this particular waveguide, single mode operation is possible from 6.562 GHz to 13.124 GHz. ^{ideally}

... This is assuming no loss. Metal losses ^{if small} tend to "smear out" the response such as



There is another characteristic of wave propagation in hollow metallic waveguides that is very apparent from these plots of β_{zmn} . These waveguides are dispersive, even when there are no losses! (Metallic losses would add further dispersion.)

That is, $\omega \propto e^{-j\beta_{zmn}z}$ propagation, the phase velocity is dependent on frequency. By definition, the phase velocity v_p is given as

$$v_p = \frac{\omega}{\beta}$$

For propagation in $+z$, the phase velocity in that direction is

$$v_{pz} = \frac{\omega}{\beta_{zmn}} = \frac{\omega}{\sqrt{\beta^2 - \beta_{cmn}^2}} \quad \beta^2 > \beta_{cmn}^2 \quad (9)$$

... Unlike plane waves in lossless media, v_{pz} is a fct. of frequency.

... Similarly, the group velocity is also a function of f . This velocity is defined as the inverse slope of the dispersion curve ($\beta_z - \omega$):

$$v_{gz} = \frac{\partial \omega}{\partial \beta_{zmn}}$$

... Group velocity \rightarrow speed of ^(energy) information transmission

... interesting to plot the phase & group velocities: (Fig 2-2):

