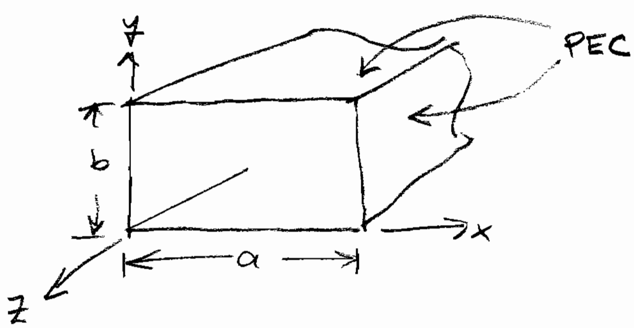
Lecture 25 - Rectangular Metallic Waveguides: TE^z & TM^z Modes.

Transmission lines are not the only way to guide EM signals. There are other structures that can accomplish this and have advantages (& some disad's.) compared to TLs. Examples of such structures are hollow metallic pipes, dielectric fibers, dielectric slabs. The latter two don't even use metal for guiding the wave!

We'll begin this discussion w/ the analysis of the rectangular metallic waveguide (Fig 8-1):



[Insert "Types of EM Waves"]

Use superposition of fields to simplify sol'n process:

(i) $E_z = 0$ and $H_z \neq 0$. Called TE^z modes.

(ii) $E_z \neq 0$ and $H_z = 0$. Called TM^z modes.

(iii) $E_z = 0$ and $H_z = 0$. Leads to indeterminant forms in field equations. TEM modes. These not physically possible in hollow metallic wvgds.

TE^z Modes

With $E_z = 0$ & $H_z \neq 0$, then the transverse field equations (7)-(10) become

$e^{-j\beta_z z}$

$$H_x = \frac{-j\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial x} \quad (11)$$

$$H_y = \frac{-j\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial y} \quad (12)$$

$$E_x = \frac{-j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial y} \quad (13)$$

$\beta_c^2 = \beta^2 - \beta_z^2$

$$E_y = \frac{j\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x} \quad (14)$$

Once we've analytically solved for H_z, we can use (11)-(14) and $E_z = 0$ to solve for all the other field components!

How do determine H_z? In Lecture 3, we derived the wave eqn

$$\nabla^2 \bar{H} + \beta^2 \bar{H} = 0 \quad (15)$$

in a lossless, source free region, where

$$\beta^2 = \omega^2 \mu \epsilon \quad (16)$$

In Cartesian coordinate system, (15) separates into

$$\nabla^2 H_x + \beta^2 H_x = 0$$
$$\nabla^2 H_y + \beta^2 H_y = 0$$
$$\nabla^2 H_z + \beta^2 H_z = 0 \quad (17)$$

The one of interest to us is (17). If we can analytically determine H_z from (17) then well on our way.

We'll define $H_z(x,y,z) = h_z(x,y)\, e^{-j\beta_z z}$ (18)

Use separation of variables for h_z as in lecture 3 we can determine that

$$h_z(x,y) = \left[A\cos\beta_x x + B\sin\beta_x x\right]\cdot\left[C\cos\beta_y y + D\sin\beta_y y\right] \quad (19)$$

and $\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$ (20)

That's it! We now have an analytical sol'n for H_z!
To evaluate the constants in (19), we need to apply the boundary conditions. With PEC walls, $\bar{E}_{tan} = 0$ on all four walls.

For TE^z modes, this means

- $E_x = 0$ for $0 \le x \le a$ at $y = 0$ and b (21)
- $E_y = 0$ for $0 \le y \le b$ at $x = 0$ and a (22)

Use (13) & (14) to determine E_x & E_y from H_z:

$$e_x(x,y) = \frac{-j\omega\mu}{\beta_c^2}\frac{\partial h_z}{\partial y} = \frac{-j\omega\mu}{\beta_c^2}\left[A\cos\beta_x x + B\sin\beta_x x\right]\cdot \quad (23)$$

$$\left[-C\beta_y\sin\beta_y y + D\beta_y\cos\beta_y y\right]$$

and

$$e_y(x,y) = \frac{j\omega\mu}{\beta_c^2}\frac{\partial h_z}{\partial x} = \frac{j\omega\mu}{\beta_c^2}\left[-A\beta_x\sin\beta_x x + B\beta_x\cos\beta_x x\right]\cdot \quad (24)$$

$$\left[C\cos\beta_y y + D\sin\beta_y y\right]$$

Apply (21):

At $y=0 \Rightarrow e_x(x,0) = \frac{-j\omega\mu}{\beta_c^2}\left[A\cos\beta_x x + B\sin\beta_x x\right]\cdot D\beta_y = 0$

Requires $D=0$.

At $y=b$ & $D=0 \Rightarrow e_x(x,b) = \frac{-j\omega\mu}{\beta_c^2}\left[A\cos\beta_x x + B\sin\beta_x x\right]\left[-C\beta_y\sin(\beta_y b)\right] = 0$

For non trivial requires

$$\beta_y b = n\pi \quad , \quad n = 0, 1, 2, \ldots$$

or

$$\beta_y = \frac{n\pi}{b} \quad n = 0, 1, 2, \cdots \qquad (25)$$

Apply (22):

At $x=0 \Rightarrow e_y(0,y) = \frac{j\omega\mu}{\beta_c^2} B\beta_x \cdot \left[C\cos\beta_y y + \cancel{D}^{\,0}\sin\beta_y y\right] = 0$

Requires $B=0$.

At $x=a$ & $B=0$ & $D=0$ $\Rightarrow e_y(a,y) = \frac{j\omega\mu}{\beta_c^2}\left[-A\beta_x\sin\beta_x a\right]\cdot\left[C\cos\beta_y y\right] = 0$

Requires for nontrivial solutions that

$$\beta_x a = m\pi$$

or

$$\beta_x = \frac{m\pi}{a} \quad m = 0, 1, 2, \cdots \qquad (26)$$

Incorporating all these results into (18) & (19) gives

$$H_z(x,y,z) = A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_{zmn}z} \qquad (27)$$

A_{mn} is a single constant that arose by "absorbing" the two constants $A \cdot C$ in (19) into a single one. The subscript indicates its possible this constant different depending on m,n / on some and the indices m & n.

The other field components are $E_z = 0$ and

$$E_x = \frac{j\omega\mu}{\beta_c^2} \cdot \frac{n\pi}{b} A_{mn} \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_{z,mn} z} \qquad (28)$$

$$E_y = -\frac{j\omega\mu}{\beta_c^2} \frac{m\pi}{a} A_{mn} \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_{z,mn} z} \qquad (29)$$

$$H_x = \frac{j\beta_{z,mn}}{\beta_c^2} \frac{m\pi}{a} A_{mn} \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_{z,mn} z} \qquad (30)$$

$$H_y = \frac{j\beta_{z,mn}}{\beta_c^2} \frac{n\pi}{b} A_{mn} \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_{z,mn} z} \qquad (31)$$

TM^z Modes

- State results
- Make hmwk prob.

The derivation of the field solutions for these modes directly parallels that for TE^z modes. (It would be nice to just use duality rather than repeat the derivation. Why is that not accurate here?)

For TM^z modes, $E_z \neq 0$ while $H_z = 0$. Applying separation of variables to the wave eqn

$$\nabla^2 E_z + \beta^2 E_z = 0$$

gives $E_z(x,y,z) = e_z(x,y) e^{-j\beta_z z}$

$$= (A \cos\beta_x x + B \sin\beta_x x)(C \cos\beta_y y + D \sin\beta_y y) e^{-j\beta_z z} \qquad (32)$$

We can apply the b.c.'s directly w/ $e_z(x,y)$. leads to

$$E_z(x,y,z) = B_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_{z,mn}z} \quad \begin{matrix} m=1,2,\cdots \\ n=1,2,\cdots \end{matrix} \tag{33}$$

Note that either $m=0$ or $n=0$ leads to a trivial soln.

The transverse field components can be found from (7)-(10) using (33) & $H_z=0$.

Lecture 25: TEM, TE and TM Modes for Waveguides. Rectangular Waveguide

Types of EM Waves

We will first develop an extremely interesting property of EM waves that propagate along homogeneous waveguides. This will lead to the concept of "modes" and their classification as

- Transverse Electric and Magnetic (TEM),
- Transverse Electric (TE), or
- Transverse Magnetic (TM).

Proceeding, from the Maxwell curl equations:

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \quad \Rightarrow \quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\bar{H}$$

or

$$\hat{x}: \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$\hat{y}: \quad -\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = -j\omega\mu H_y$$

$$\hat{z}: \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

Assuming there is wave propagation in the $+z$ direction, the spatial variation in z is known so that

$$\frac{\partial\left(e^{-j\beta_z z}\right)}{\partial z} = -j\beta_z\left(e^{-j\beta_z z}\right)$$

Consequently, these curl equations simplify to

$$\frac{\partial E_z}{\partial y} + j\beta_z E_y = -j\omega\mu H_x \tag{1}$$

$$-\frac{\partial E_z}{\partial x} - j\beta_z E_x = -j\omega\mu H_y \tag{2}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \tag{3}$$

We can perform a similar expansion of Ampère's equation $\nabla \times \bar{H} = j\omega\varepsilon\bar{E}$ to obtain

$$\frac{\partial H_z}{\partial y} + j\beta_z H_y = j\omega\varepsilon E_x \tag{4}$$

$$-j\beta_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \tag{5}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \tag{6}$$

Now, (1)-(6) can be manipulated to produce simple algebraic equations for the transverse (x and y) components of $\bar{E}$ and $\bar{H}$. For example, from (1):

$$H_x = \frac{j}{\omega\mu}\left(\frac{\partial E_z}{\partial y} + j\beta_z E_y\right)$$

Substituting for E_y from (5) we find

$$H_y = -\frac{j}{\beta_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right)$$

or,

$$\boxed{H_x = \frac{j}{\beta_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right)} \qquad (7)$$

where $\boxed{\beta_c^2 \equiv \beta^2 - \beta_z^2}$ and $\beta^2 = \omega^2\mu\varepsilon$.

Similarly, we can show that

$$\boxed{H_y = -\frac{j}{\beta_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right)} \qquad (8)$$

$$\boxed{E_x = \frac{-j}{\beta_c^2}\left(\beta_z\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right)} \qquad (9)$$

$$\boxed{E_y = \frac{j}{\beta_c^2}\left(-\beta_z\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right)} \qquad (10)$$

Most important point: From (7)-(10), we can see that **all transverse** components of $\bar{E}$ and $\bar{H}$ can be determined from **only the axial** components E_z and H_z. It is this fact that allows the mode designations TEM, TE, and TM.

Furthermore, we can use superposition to reduce the complexity of the solution by considering each of these mode types separately, then adding the fields together at the end.