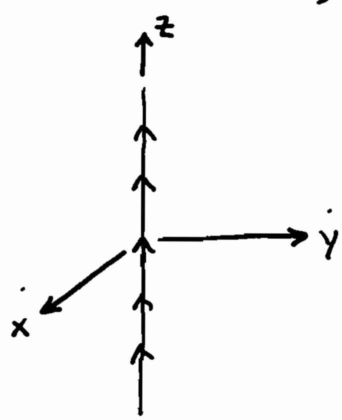
Green's Function for 2-D Sources

The Green's function for a given problem, as we've seen before, is simply the unit impulse response of the space. The "source" is a delta function using the smallest unit of excitation, for the space.

For the 3-D homogeneous space, the source was a pt. dipole. For the <u>2-D problem</u>, the source of cylindrical waves is a <u>line source</u>.

Consider a line source (z directed) invariant along the z axis –



Let $\quad \bar{J}(z) = \hat{z}\, 1 \cdot \delta(x)\delta(y)$

From the vector potential method, it is known that

$$\nabla^2 \bar{A} + k^2 \bar{A} = -\bar{J}$$

Away from the z axis, $\bar{A}$ must satisfy the differential eqn –

$$\nabla^2 \bar{A} + k^2 \bar{A} = 0 \qquad \text{(homogeneous)}$$

From the separation of variables technique, the z component of $\bar{A}$ will have the form

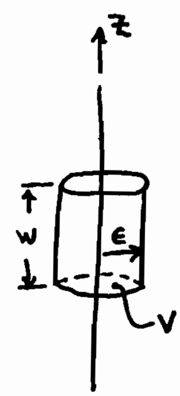
$$A_z = f(\rho)\, g(\phi)\, h(z) \qquad \text{(cylindrical coordinates)}$$

By symmetry, A_z is independent of both ϕ and z. This will be satisfied by choosing $\underline{n=0}$ and $\underline{k_z=0}$.

What about the ρ variation in $f(\rho)$? Since only outgoing waves are to be expected, we'll choose the Hankel function, 2nd Kind. Therefore –

$$\underline{\underline{A_z(\rho) = \underset{\text{constant}}{C}\, H^{(2)}_{\underset{n=0}{0}}(k\rho)}}$$

In order to evaluate the constant C, the boundary conditions must be applied at the source. Starting with $\nabla^2\bar{A} + k^2\bar{A} = -\bar{J}$ and integrating over the volume V as shown –



gives –

$$\int_V \left[\nabla^2\bar{A} + k^2\bar{A}\right] dv = -\int_V \delta(x)\delta(y)\, dv = -\int_z dz$$

In the limit as $\epsilon \to 0$, the $k^2 A_z$ term becomes much smaller than the $k^2 A_z$ term. That is, we approach the magnetostatic regime. This leaves

$$\lim_{\epsilon \to 0} \left\{ \int_V \nabla^2 A_z \, dv = -\int_z 1 \, dz \right\}$$

but,

$$\int_V \nabla^2 A_z \, dv = \int_V \nabla \cdot \nabla \bar{A}_z \, dv \underset{\uparrow}{=} \oint_{S(V)} \nabla A_z \cdot \hat{n} \, ds$$

Divergence Theorem

Therefore,

$$\lim_{\epsilon \to 0} \left\{ \int_z \int_0^{2\pi} \frac{\partial A_z}{\partial \rho} \rho \, d\phi \, dz = -\int_z dz \right\}$$

ignoring integrals over endcaps which cancel each other.

OR,

$$\lim_{\epsilon \to 0} \left\{ 2\pi\epsilon \frac{\partial A_z}{\partial \rho} = -1 \right\} \qquad (1)$$

Remembering that $A_z = C H_0^{(2)}(k\rho)$, what is $\frac{\partial A_z}{\partial \rho}$?

Considering $\frac{\partial H_0^{(2)}(k\rho)}{\partial \rho}$ and the change of variables

$$v = k\rho \Rightarrow \rho = \frac{v}{k}$$

then

$$\frac{d H_0^{(2)}(k\rho)}{d\rho} = \frac{dv}{d\rho} \frac{d H_0^{(2)}(v)}{dv} = k \frac{d H_0^{(2)}(v)}{dv}$$

Derivatives of Bessel functions are related as

$$B_n'(x) = -B_{n+1}(x) + \frac{n}{x} B_n(x)$$

where $\quad B_n(x) = J_n(x),\ N_n(x),\ H_n^{(1)}(x),\ H_n^{(2)}(x), \cdots$

For the special case when $n=0 \Rightarrow \quad B_n'(x) = -B_{n+1}(x)$

Therefore, $$\frac{dH_0^{(2)}(v)}{dv} = -H_1^{(2)}(v)$$

and $$\frac{dH_0^{(2)}(k\rho)}{d\rho} = -k H_1^{(2)}(k\rho)$$

However, as $k\rho \to 0$ as in our case

$$\lim_{k\rho\to 0}\left\{-k H_1^{(2)}(k\rho)\right\} \approx -k\Big[\underbrace{\frac{k\rho}{2}}_{\to 0} + j\frac{2}{\pi k\rho}\Big]$$

since small w.r.t.

$$\approx -j\frac{2}{\pi\rho} = -\frac{j2}{\pi\epsilon}$$

Substituting this result into ① —

$$\lim_{\epsilon\to 0}\left\{2\pi\epsilon\left(\frac{-j2}{\pi\epsilon}\right) C = -1\right\}$$

$$\Rightarrow \quad \underline{\underline{C = \frac{1}{4j}}}$$

which finally results in

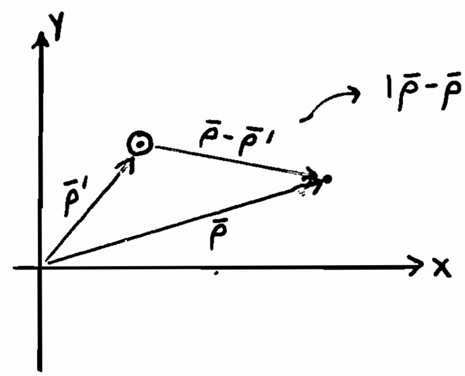
$$A_z(\rho) = \frac{1}{4j} H_0^{(2)}(k\rho) \qquad (2)$$

from unit-amplitude line current source!

For a line source displaced from the z axis in a homogeneous space, the substitution

$$\rho \rightarrow |\bar{\rho} - \bar{\rho}'|$$

where



$|\bar{\rho}-\bar{\rho}'|$ = distance from source point to a.b. pt. in the x-y plane.

into (2) results in

$$A_z(\bar{\rho}) = \frac{1}{4j} H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)$$

whereby, by definition, the Green's fct. for a 2-D homogeneous space is

$$\underline{\underline{g(\bar{\rho}|\bar{\rho}') = \frac{1}{4j} H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)}}$$

Using this Green's function, we can find the $\bar{E}$ & $\bar{H}$ fields produced by a line current of amplitude I_o located at $\rho'=0$.

with $\bar{H} = \nabla \times \bar{A}$

$$= \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \cancel{\frac{\partial}{\partial\phi}}^{0} & \cancel{\frac{\partial}{\partial z}}^{0} \\ \cancel{A_\rho}^{0} & \cancel{\rho A_\phi}^{0} & A_z \end{vmatrix} = -\frac{\rho}{\rho}\hat{\phi}\frac{\partial A_z}{\partial\rho} = -\hat{\phi}\frac{I_0}{4j}\left(-kH_1^{(2)}(k\rho)\right)$$

or,

$$\underline{\bar{H}(\rho) = \hat{\phi}\,\frac{kI_0}{4j}\,H_1^{(2)}(k\rho)}$$

With $\bar{E} = -j\frac{\eta}{k}\nabla\times\bar{H} = -j\frac{\eta}{k}\frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \cancel{\frac{\partial}{\partial\phi}}^{0} & \cancel{\frac{\partial}{\partial z}}^{0} \\ \cancel{H_\rho}^{0} & \rho H_\phi & \cancel{H_z}^{0} \end{vmatrix}$

$$= -j\frac{\eta}{k}\hat{z}\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial\rho} = -j\frac{\eta}{k}\hat{z}\left[\frac{1}{\rho}\rho\frac{\partial H_\phi}{\partial\rho} + \frac{1}{\rho}\right]$$

then substituting for H_ϕ from above gives

$$\bar{E} = -\hat{z}\,\frac{j\eta}{k}\left(\frac{kI_0}{4j}\right)\left[\frac{\partial H_1^{(2)}(k\rho)}{\partial\rho} + \frac{1}{\rho}H_1^{(2)}(k\rho)\right]$$

But $B_n'(x) = B_{n-1}(x) - \frac{n}{x}B_n(x)$

or in this case $\rightarrow$ $\dfrac{dH_1^{(2)}(v)}{dv} = H_0^{(2)} - \frac{1}{v}H_1^{(2)}(v)$

such that $\dfrac{k}{k}\dfrac{dH_1^{(2)}(k\rho)}{d\rho} = k\dfrac{dH_1^{(2)}(v)}{dv} = k\left[H_0^{(2)}(v) - \frac{1}{v}H_1^{(2)}(v)\right]$

$$= k\,H_0^{(2)}(k\rho) - \frac{1}{\rho}H_1^{(2)}(k\rho)$$

Sostituting into our expression for $\bar{E}$ gives

$$\bar{E}(\rho) = -\hat{z}\,\frac{\eta I_o}{4}\left[k H_o^{(2)}(k\rho) - \frac{1}{\rho}\cancel{H_1^{(2)}(k\rho)} + \frac{1}{\rho}\cancel{H_1^{(2)}(k\rho)}\right]$$

or

$$\underline{\bar{E}(\rho) = -\hat{z}\, I_o\,\frac{k\eta}{4}\, H_o^{(2)}(k\rho)}$$

In the far-field as $k\rho \to \infty$ & using the asymptotic forms of the Hankel functions -

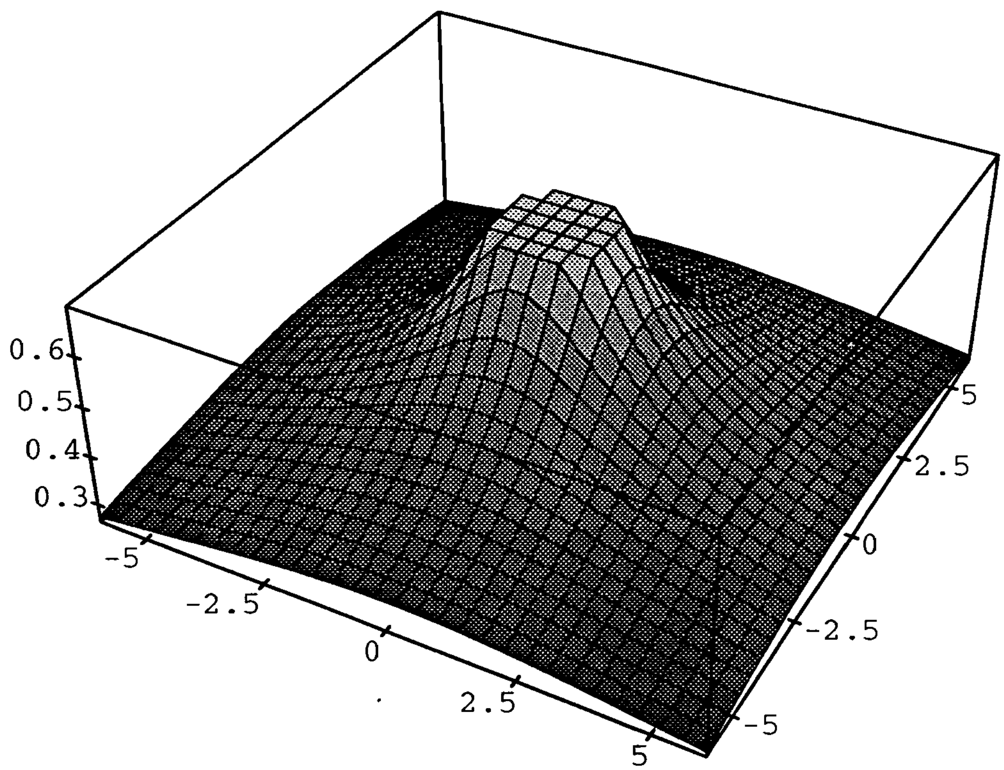
$$H_\phi \sim k I_o \sqrt{\frac{j}{8\pi k\rho}}\; e^{-jk\rho}$$

$$E_z \sim -k\eta I_o \sqrt{\frac{j}{8\pi k\rho}}\; e^{-jk\rho}.$$

Notice:

- The fields decay as $\frac{1}{\sqrt{\rho}}$, in contrast to $\frac{1}{r}$ decay in 3-D
- $\frac{E_z}{H_\phi} \sim -\eta$: fields are cylindrical waves $e^{-jk\rho}$ in far zone.

$H_0^{(2)}(k\rho)$ Magnitude



$H_0^{(2)}(-k\rho)$ Phase (radians)