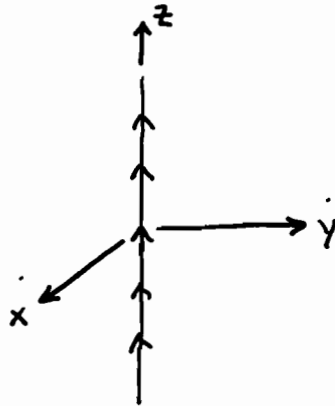


The Green's function for a given problem, as we've seen before, is simply the unit impulse response of the space. The "source" is a delta function using the smallest unit of excitation, for the space.

For the 3-D homogeneous space, the source was a pt. dipole.

For the 2-D problem, the source of cylindrical waves is a line source.

Consider a line source (z directed) invariant along the z axis -



$$\text{Let } \vec{J}(z) = \hat{z} I \cdot \delta(x) \delta(y)$$

From the vector potential method, it is known that

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\vec{J}$$

Away from the z axis, \vec{A} must satisfy the differential eqn -

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0 \quad (\text{homogeneous})$$

From the separation of variables technique, the z component of \bar{A} will have the form

$$A_z = f(\rho) g(\phi) h(z) \quad (\text{cylindrical coordinates})$$

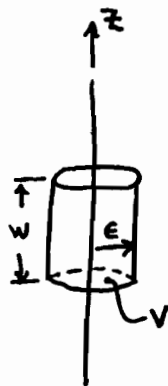
By symmetry, A_z is independent of both ϕ and z . This will be satisfied by choosing $n=0$ and $k_z=0$.

What about the ρ variation in $f(\rho)$? Since only outgoing waves are to be expected, we'll choose the Hankel function, 2nd kind. Therefore -

$$\underline{A_z(\rho) = C H_0^{(2)}(k\rho)}$$

\uparrow constant $n=0$

In order to evaluate the constant C , the boundary conditions must be applied at the source. Starting with $\nabla^2 \bar{A} + k^2 \bar{A} = -\bar{J}$ and integrating over the volume V as shown -



gives -

$$\int_V [\nabla^2 \bar{A} + k^2 \bar{A}] dV = - \int_V \delta(x)\delta(y) dV = - \int_z dz$$

In the limit as $\epsilon \rightarrow 0$, the $k^2 A_z$ term becomes much smaller than the $k^2 A_z$ term. That is, we approach the magnetostatic regime. This leaves

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_V \nabla^2 A_z \, dV = - \int_z 1 \, dz \right\}$$

but,

$$\int_V \nabla^2 A_z \, dV = \int_V \nabla \cdot \nabla \bar{A}_z \, dV = \oint_{S(V)} \nabla A_z \cdot \hat{n} \, ds$$

↑
Divergence Theorem

Therefore,

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_z \int_0^{2\pi} \frac{\partial A_z}{\partial \rho} \rho \, d\phi \, dz = - \int_z dz \right\}$$

ignoring integrals over endcaps which cancel each other.

$$\text{OR,} \quad \lim_{\epsilon \rightarrow 0} \left\{ 2\pi\epsilon \frac{\partial A_z}{\partial \rho} = -1 \right\} \quad (1)$$

Remembering that $A_z = C H_0^{(z)}(kr)$, what is $\frac{\partial A_z}{\partial \rho}$?

Considering $\frac{\partial H_0^{(z)}(kr)}{\partial \rho}$ and the change of variables

$$v = kr \Rightarrow \rho = \frac{v}{k}$$

$$\text{then} \quad \frac{dH_0^{(z)}(kr)}{d\rho} = \frac{dv}{d\rho} \frac{dH_0^{(z)}(v)}{dv} = k \frac{dH_0^{(z)}(v)}{dv}$$

Derivatives of Bessel functions are related as

$$B'_n(x) = -B_{n+1}(x) + \frac{n}{x} B_n(x)$$

where $B_n(x) = J_n(x), N_n(x), H_n^{(1)}(x), H_n^{(2)}(x), \dots$

For the special case when $n=0 \Rightarrow B'_n(x) = -B_{n+1}(x)$

Therefore,
$$\frac{dH_0^{(2)}(v)}{dv} = -H_1^{(2)}(v)$$

and
$$\frac{dH_0^{(2)}(kp)}{dp} = -k H_1^{(2)}(kp)$$

However, as $kp \rightarrow 0$ as in our case

$$\begin{aligned} \lim_{kp \rightarrow 0} \left\{ -k H_1^{(2)}(kp) \right\} &\approx -k \left[\underbrace{\frac{kp}{z}}_0 + j \frac{z}{\pi kp} \right] \\ &\quad \text{since small w.r.t.} \\ &\approx -j \frac{z}{\pi p} = -\frac{jz}{\pi \epsilon} \end{aligned}$$

Substituting this result into ① -

$$\lim_{\epsilon \rightarrow 0} \left\{ 2\pi \epsilon \left(\frac{-jz}{\pi \epsilon} \right) C = -1 \right\}$$

$$\Rightarrow \underline{\underline{C = \frac{1}{4j}}}$$

which finally results in

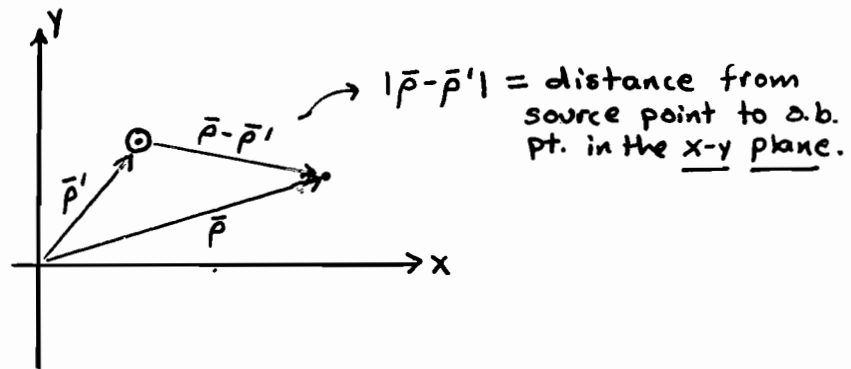
$$A_z(\rho) = \frac{1}{4j} H_0^{(2)}(k\rho) \quad \textcircled{2}$$

from unit-amplitude line current source!

For a line source displaced from the z axis in a homogeneous space, the substitution

$$\rho \rightarrow |\bar{\rho} - \bar{\rho}'|$$

where



into $\textcircled{2}$ results in

$$A_z(\bar{\rho}) = \frac{1}{4j} H_0^{(2)}(k|\bar{\rho} - \bar{\rho}'|)$$

whereby, by definition, the Green's fct. for a 2-D homogeneous space is

$$\underline{\underline{g(\bar{\rho}|\bar{\rho}') = \frac{1}{4j} H_0^{(2)}(k|\bar{\rho} - \bar{\rho}'|)}}$$

Using this Green's function, we can find the \bar{E} & \bar{H} fields produced by a line current of amplitude I_0 located at $\rho' = 0$.

with $\bar{H} = \nabla \times \bar{A}$

$$= \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_{\rho} & \rho A_{\phi} & A_z \end{vmatrix} = -\frac{\rho}{\rho} \hat{\phi} \frac{\partial A_z}{\partial \rho} = -\hat{\phi} \frac{I_0}{4j} [-k H_1^{(2)}(kp)]$$

or,

$$\bar{H}(\rho) = \hat{\phi} \frac{k I_0}{4j} H_1^{(2)}(kp)$$

$$\begin{aligned} \text{With } \bar{E} &= -j \frac{1}{k} \nabla \times \bar{H} = -j \frac{1}{k} \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_{\rho} & \rho H_{\phi} & H_z \end{vmatrix} \\ &= -j \frac{1}{k} \hat{z} \frac{1}{\rho} \frac{\partial (\rho H_{\phi})}{\partial \rho} = -j \frac{1}{k} \hat{z} \left[\frac{1}{\rho} \rho \frac{\partial H_{\phi}}{\partial \rho} + \frac{1}{\rho} \right] \end{aligned}$$

then substituting for H_{ϕ} from above gives

$$\bar{E} = -\hat{z} \frac{j\eta}{k} \left(\frac{k I_0}{4j} \right) \left[\frac{\partial H_1^{(2)}(kp)}{\partial \rho} + \frac{1}{\rho} H_1^{(2)}(kp) \right]$$

$$\text{But } B_n'(x) = B_{n-1}(x) - \frac{n}{x} B_n(x)$$

$$\text{or in this case } \rightarrow \frac{d H_1^{(2)}(v)}{dv} = H_0^{(2)} - \frac{1}{v} H_1^{(2)}(v)$$

$$\begin{aligned} \text{such that } \frac{k}{k} \frac{d H_1^{(2)}(kp)}{d\rho} &= k \frac{d H_1^{(2)}(v)}{dv} = k \left[H_0^{(2)}(v) - \frac{1}{v} H_1^{(2)}(v) \right] \\ &= k H_0^{(2)}(kp) - \frac{1}{\rho} H_1^{(2)}(kp) \end{aligned}$$

Substituting into our expression for \bar{E} gives

$$\bar{E}(\rho) = -\hat{z} \frac{\gamma I_0}{4} \left[k H_0^{(2)}(k\rho) - \frac{1}{\rho} H_1^{(2)}(k\rho) + \frac{1}{\rho} H_1^{(2)}(k\rho) \right]$$

or

$$\bar{E}(\rho) = -\hat{z} I_0 \frac{k\gamma}{4} H_0^{(2)}(k\rho)$$

In the far-field as $k\rho \rightarrow \infty$ using the asymptotic forms of the Hankel functions -

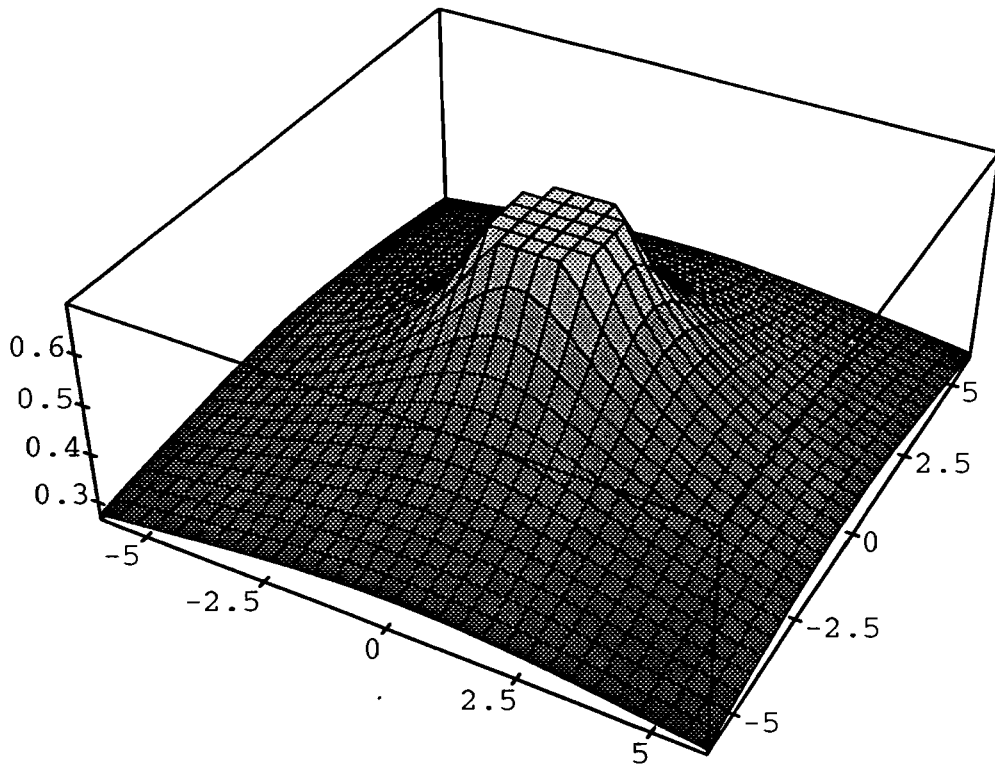
$$H_\phi \sim k I_0 \sqrt{\frac{j}{8\pi k\rho}} e^{-jk\rho}$$

$$E_z \sim -k\gamma I_0 \sqrt{\frac{j}{8\pi k\rho}} e^{-jk\rho}$$

Notice: • The fields decay as $\frac{1}{\sqrt{\rho}}$, in contrast to $\frac{1}{r}$ decay in 3-D

• $\frac{E_z}{H_\phi} \sim -\gamma$: fields are cylindrical waves $e^{-jk\rho}$ in far zone.

$H_0^{(z)}$ (kp) Magnitude



$H_D^{(z)}(kp)$ Phase (radians)

