The Green's function for a given problem, as we've seen before, is simply the unit impulse response of the space. The "source" is a delta function using the smallest unit of excitation, for the space.

For the 3-D homogeneous space, the source was a pt. dipole. For the 2-D problem, the source of cylindrical waves is a line source.

Consider a line source ($z$ directed) invariant along the $z$ axis—

Let $\vec{J}(z) = \hat{z} \cdot \delta(x) \delta(y)$

From the vector potential method, it is known that

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\vec{J}$$

Away from the $z$ axis, $\vec{A}$ must satisfy the differential eqn—

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0 \quad (\text{homogeneous})$$
From the separation of variables technique, the \( \tau \) component of \( \vec{A} \) will have the form

\[
A_\tau = f(\rho) \varphi(\phi) h(\tau) \quad \text{(cylindrical coordinates)}
\]

By symmetry, \( A_\tau \) is independent of both \( \phi \) and \( \tau \). This will be satisfied by choosing \( n=0 \) and \( \frac{\partial}{\partial \tau} = 0 \).

What about the \( \rho \) variation in \( f(\rho) \)? Since only outgoing waves are to be expected, we'll choose the Hankel function, \( 2^{nd} \) kind. Therefore:

\[
A_\tau(\rho) = C H_0^{(2)}(k \rho)
\]

In order to evaluate the constant \( C \), the boundary conditions must be applied at the source. Starting with \( \nabla^2 \vec{A} + k^2 \vec{A} = -\vec{J} \) and integrating over the volume \( V \) as shown:

\[
\int_V \left[ \nabla^2 \vec{A} + k^2 \vec{A} \right] \, dv = -\int_V J(x) \delta(y) \, dv = -\int_z d\tau
\]
In the limit as $e \to 0$, the $k^2 A_z$ term becomes much smaller than the $k^2 A_z$ term. That is, we approach the magnetostatic regime. This leaves

$$
\lim_{e \to 0} \left\{ \int_\Omega \nabla^2 A_z \, dv = -\int_\Omega 1 \, dz \right\}
$$

but,

$$
\int_\Omega \nabla^2 A_z \, dv = \int_\Omega \nabla \cdot \nabla A_z \, dv = \oint_{\partial \Omega} \nabla A_z \cdot \hat{n} \, ds \quad \text{Divergence Theorem}
$$

Therefore,

$$
\lim_{e \to 0} \left\{ \int_\Omega \frac{\partial A_z}{\partial \rho} \rho \, d\phi \, dz = -\int_\Omega 1 \, dz \right\}
$$

ignoring integrals over endcaps which cancel each other.

or,

$$
\lim_{e \to 0} \left\{ 2\pi e \frac{\partial A_z}{\partial \rho} = -1 \right\} \quad (1)
$$

Remembering that $A_z = C H_{2}^{(1)}(kp)$, what is $\frac{\partial A_z}{\partial \rho}$?

Considering $\frac{\partial H_{2}^{(1)}(kp)}{\partial \rho}$ and the change of variables

$$
v = kp \Rightarrow \rho = \frac{v}{k}
$$

then

$$
\frac{d H_{2}^{(1)}(kp)}{dp} = \frac{dv}{dp} \frac{d H_{2}^{(1)}(v)}{dv} = k \frac{d H_{2}^{(1)}(v)}{dv}
$$
Derivatives of Bessel functions are related as

\[ B'_n(x) = -B_{n+1}(x) + \frac{n}{x} B_n(x) \]

where \( B_n(x) = J_n(x), N_n(x), H_n^{(1)}(x), H_n^{(2)}(x), \ldots \)

For the special case when \( n=0 \) \( \Rightarrow \ B'_0(x) = -B_{n+1}(x) \)

Therefore, \[ \frac{dH_0^{(2)}(\nu)}{d\nu} = -H_1^{(2)}(\nu) \]

and \[ \frac{dH_0^{(3)}(\kappa \rho)}{d\rho} = -\kappa H_1^{(2)}(\kappa \rho) \]

However, as \( \kappa \rho \to 0 \) as in our case

\[ \lim_{\kappa \rho \to 0} \left\{ -\kappa H_1^{(2)}(\kappa \rho) \right\} \approx -\kappa \left[ \frac{\kappa \rho}{\nu_0^2} + j \frac{2}{\pi \kappa \rho} \right] \text{ since small w.r.t.} \]

\[ \approx -j \frac{2}{\pi \rho} = -j \frac{2}{\pi \epsilon} \]

Substituting this result into (1) -

\[ \lim_{\epsilon \to 0} \left\{ 2\pi \epsilon \left( -\frac{j2}{\pi \epsilon} \right) C = -1 \right\} \]

\[ \Rightarrow \quad C = \frac{1}{4j} \]
which finally results in

\[ A_z(\rho) = \frac{1}{4j} H_0^{(2)}(k\rho) \]  \hspace{1cm} (2)

from unit-amplitude line current source!

For a line source displaced from the z axis in a homogeneous space, the substitution

\[ \rho \rightarrow |\bar{\rho} - \bar{\rho}'| \]

where

\[ |\bar{\rho} - \bar{\rho}'| = \text{distance from source point to pt. in the x-y plane} \]

into (2) results in

\[ A_z(\bar{\rho}) = \frac{1}{4j} H_0^{(2)}(k|\bar{\rho} - \bar{\rho}'|) \]

whereby, by definition, the Green's funct. for a 2-D homogeneous space is

\[ g(\bar{\rho} | \bar{\rho}') = \frac{1}{4j} H_0^{(2)}(k|\bar{\rho} - \bar{\rho}'|) \]

Using this Green's function, we can find the \( E \) and \( \nabla \times \mathbf{H} \) fields produced by a line current of amplitude \( I_0 \) located at \( \rho' = 0 \).
with \( \mathbf{H} = \nabla \times \mathbf{A} \)
\[
\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{vmatrix} = -\frac{\rho}{\rho} \frac{\partial A_z}{\partial \rho} = -\frac{\rho}{4j} \left( -k H_i^{(2)}(kr) \right)
\]

or,
\[
\mathbf{H}(r) = \hat{r} \frac{kI_0}{4j} H_i^{(2)}(kr)
\]

With \( \mathbf{E} = -j\frac{\gamma}{k} \nabla \times \mathbf{H} = -j\frac{\gamma}{k} \frac{1}{\rho} \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{vmatrix} = -j\frac{\gamma}{k} \left[ \frac{1}{\rho} \frac{\partial H_\phi}{\partial \rho} + \frac{1}{\rho^2} \right]
\]

then substituting for \( H_\phi \) from above gives
\[
\mathbf{E} = -\hat{z} \frac{j\gamma}{k} \left( \frac{kI_0}{4j} \right) \left[ \frac{\partial H_i^{(2)}(kr)}{\partial \rho} + \frac{1}{\rho} H_i^{(2)}(kr) \right]
\]

But \( B_n'(x) = B_{n-1}(x) - \frac{n}{x} B_n(x) \)

or in this case \( \frac{dH_i^{(2)}(v)}{dv} = H_i^{(2)}(v) - \frac{1}{v} H_i^{(3)}(v) \)

such that \( \frac{k}{k} \frac{dH_i^{(2)}(kr)}{d\rho} = \frac{k}{k} \frac{dH_i^{(2)}(v)}{dv} = \frac{k}{k} \left[ H_i^{(2)}(v) - \frac{1}{v} H_i^{(3)}(v) \right] \)

\( = k H_i^{(2)}(kr) - \frac{1}{\rho} H_i^{(2)}(kr) \)
Sustituting into our expression for \( E \) gives

\[
E(\rho) = -\frac{\hat{z}}{4} \frac{I_0}{4} \left[ k H_0^{(2)}(kp) - \frac{1}{\rho} H_1^{(2)}(kp) + \frac{1}{\rho} \nabla \times H_0^{(2)}(kp) \right]
\]

or

\[
E(\rho) = -\frac{\hat{z}}{4} I_0 \frac{k \eta}{4} H_0^{(2)}(kp)
\]

In the far-field as \( kp \to \infty \) using the asymptotic forms of the Hankel functions

\[
H_\phi \sim k I_0 \sqrt{\frac{j}{8 \pi kp}} e^{-jkp}
\]

\[
E_z \sim -k \eta I_0 \sqrt{\frac{j}{8 \pi kp}} e^{-jkp}
\]

Notice:

- The fields decay as \( \frac{1}{\sqrt{\rho}} \), in contrast to \( \frac{1}{\rho} \) decay in 3-D

- \( \frac{E_z}{H_\phi} \sim \frac{\eta}{k} \): fields are cylindrical waves \( e^{-jkp} \) in far zone.
$H_0^{(2)}(kp) \text{ Magnitude}$
$H_0^{(2)}(k_0)$ Phase (radians)