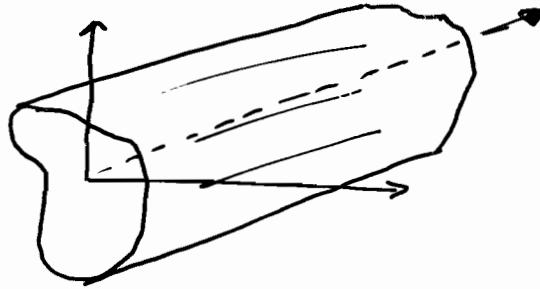


One type of geometry which receives a great deal of attention in electromagnetics is the 2-D geometry. This means the scattering body is invariant in one dimension. While the scatterer is certainly three dimensional, due to the longitudinal invariance it is referred to as 2-D.



In order to analyze these problems we must first find the form of the solutions to the wave eqn. in 2-D.

From our discussions earlier, the wave equation for time harmonic fields was found to have the form -

$$\nabla^2 \bar{T} + k^2 \bar{T} = 0 \quad (1)$$

where  $\bar{T} = \bar{E}$  or  $\bar{H}$

What we're interested in finding now are the solutions to this equation in cylindrical coordinates -

$$\bar{T} = \hat{\rho} T_\rho(\rho, \phi, z) + \hat{\phi} T_\phi(\rho, \phi, z) + \hat{z} T_z(\rho, \phi, z)$$

Note: (1) cannot be reduced to uncoupled scalar wave equations for  $T_\rho, T_\phi$  &  $T_z$  since  $\nabla^2 \bar{T}$  is defined on the Cartesian components of  $T$ .

By definition,  $\nabla \times \nabla \times \bar{T} = \nabla(\nabla \cdot \bar{T}) - \nabla^2 \bar{T}$   
 $\Rightarrow \nabla^2 \bar{T} = \nabla(\nabla \cdot \bar{T}) - \nabla \times \nabla \times \bar{T}$

$\therefore$ , the wave eqn ① becomes

$$\nabla(\nabla \cdot \bar{T}) - \nabla \times \nabla \times \bar{T} + k^2 \bar{T} = 0 \quad \text{②}$$

Expanding out the L.H.S. and noting that

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

where  $\psi = \psi(\rho, \phi, z)$  is a scalar function

give three second-order partial differential equations (PDEs):

$$\nabla^2 T_\rho + \left( -\frac{T_\rho}{\rho^2} - \frac{z}{\rho^2} \frac{\partial T_\phi}{\partial \phi} \right) + k^2 T_\rho = 0$$

$$\nabla^2 T_\phi + \left( -\frac{T_\phi}{\rho^2} + \frac{z}{\rho^2} \frac{\partial T_\rho}{\partial \rho} \right) + k^2 T_\phi = 0$$

coupled pde's.  
very difficult to solve.

$$\nabla^2 T_z + k^2 T_z = 0 \quad \text{③}$$

Since ③ is uncoupled it will be the easiest to work with.

Expanding the Laplacian in ③ and letting  $\psi = T_z$  gives

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0 \quad \text{④}$$

Following the method of Separation of Variables, we will seek solutions of the form

$$\psi = f(\rho) g(\phi) h(z) \quad \leftarrow \text{product form}$$

Substituting this product form into (4) gives -

$$gh \frac{\partial^2 f}{\partial \rho^2} + \frac{gh}{\rho} \frac{\partial f}{\partial \rho} + \frac{fh}{\rho^2} \frac{\partial^2 g}{\partial \phi^2} + fg \frac{\partial^2 h}{\partial z^2} = -k^2 fgh$$

Dividing by  $fgh = \psi$  -

$$\frac{1}{f} \frac{d^2 f}{d\rho^2} + \frac{1}{f\rho} \frac{df}{d\rho} + \frac{1}{g\rho^2} \frac{d^2 g}{d\phi^2} + \frac{1}{h} \frac{d^2 h}{dz^2} + k^2 = 0 \quad (5)$$

The last term  $\frac{1}{h} \frac{d^2 h}{dz^2}$  is not a function of  $\rho$  and  $\phi$  and must also be independent of  $z$  if the L.H.S. of (5) is to sum to zero  $\forall \rho, \phi$ .

$$\Rightarrow \frac{1}{h} \frac{d^2 h}{dz^2} = -k_z^2 \quad \text{where } k_z = \text{constant} \quad (6)$$

Substituting into (5) and after multiplication by  $\rho^2$  -

$$\frac{\rho^2}{f} \frac{d^2 f}{d\rho^2} + \frac{\rho}{f} \frac{df}{d\rho} + \frac{1}{g} \frac{d^2 g}{d\phi^2} + \rho^2 (k^2 - k_z^2) = 0 \quad (7)$$

Now, the  $\frac{1}{g} \frac{d^2 g}{d\phi^2}$  term is independent of  $\rho, z$  and also must be independent of  $\phi$  if the L.H.S. of (7) is to sum to zero  $\forall \rho, z$ .

$$\Rightarrow \frac{1}{g} \frac{d^2 g}{d\phi^2} = -n^2 \quad \text{where } n = \text{constant} \quad (8)$$

Substituting (8)  $\rightarrow$  (7) and multiplying by  $f$  gives

$$\rho^2 \frac{d^2 f}{d\rho^2} + \rho \frac{df}{d\rho} + f [-n^2 + \rho^2 (k^2 - k_z^2)] = 0$$

Defining  $k_p^2 = k^2 - k_z^2$  gives

$$\rho^2 \frac{d^2 f}{d\rho^2} + \rho \frac{df}{d\rho} + f [(k_p \rho)^2 - n^2] = 0 \quad (9)$$

This differential equation for  $f(\rho)$  is called Bessel's equation of order  $n$ .

Solutions to (9) are in the form of infinite series — not unlike other tabulated functions like trigonometric functions.

Commonly used solutions to Bessel's equation are —

$$f(\rho) = \begin{aligned} & \bullet J_n(k_p \rho) - \text{Bessel fct., order } n \\ & \bullet N_n(k_p \rho) - \text{Neuman fct., order } n \\ & \bullet H_n^{(1)}(k_p \rho) - \text{Hankel fct. of 1<sup>st</sup> kind, order } n \\ & \bullet H_n^{(2)}(k_p \rho) - \text{Hankel fct. of 2<sup>nd</sup> kind, order } n. \end{aligned}$$

Any two of these functions are linearly independent so that  $f(\rho)$  is in general a linear combination of any two of them.

$$\text{From (8) - } \frac{1}{g} \frac{d^2 g}{d\phi^2} = -n^2 \quad \text{or} \quad \frac{d^2 g}{d\phi^2} + n^2 g = 0$$

$$\text{which has solutions } g(\phi) = \begin{aligned} & \bullet \cos n\phi \\ & \bullet \sin n\phi \\ & \bullet e^{jn\phi} \\ & \bullet e^{-jn\phi} \end{aligned}$$

Note: in order for single-valued solutions ( $g(\phi) = g(\phi + 2\pi)$ ) implies  $n = \text{integer}$ .

$$\text{From (6) - } \frac{1}{h} \frac{d^2 h}{dz^2} = -k_z^2 \quad \text{or} \quad \frac{d^2 h}{dz^2} + k_z^2 h = 0$$

which has solutions  $h(z) =$

- $\cos(k_z z)$
- $\sin(k_z z)$
- $e^{jk_z z}$
- $e^{-jk_z z}$

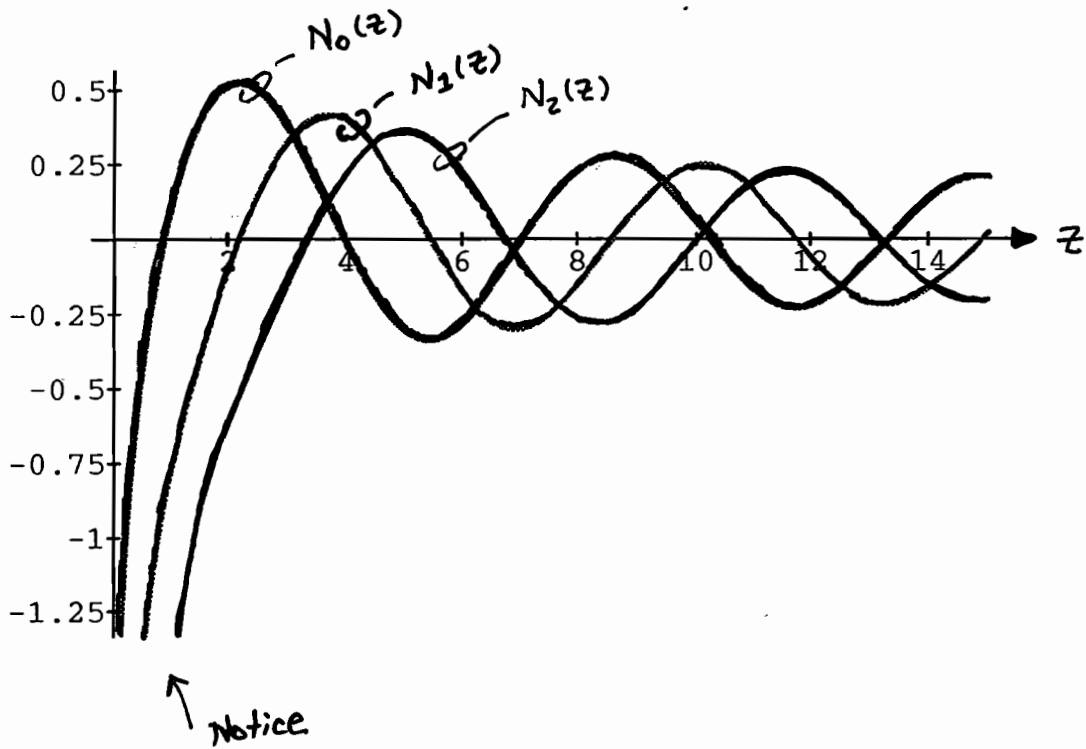
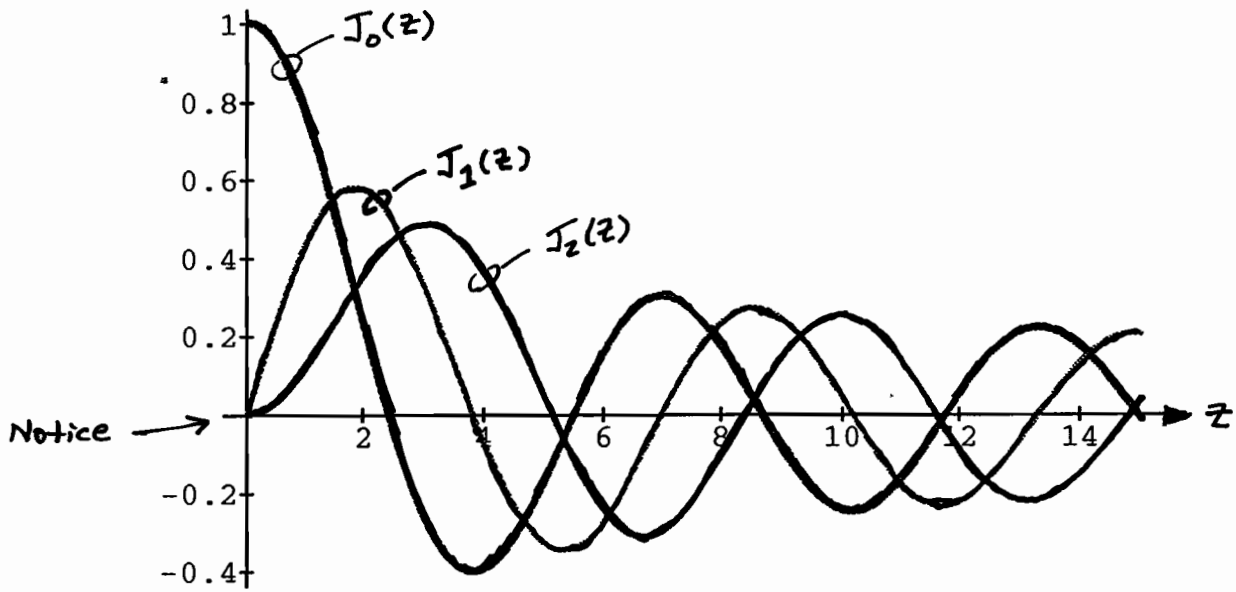
The specific linear combinations used for  $f(\rho)$ ,  $g(\phi)$  and  $h(z)$  depend a great deal on the type of problem which is to be solved.

For example, in a cylindrical waveguide, propagating (or evanescent) modes would be adequately represented by an  $e^{\pm jk_z z}$  dependence. In a resonant cavity would likely used  $\cos k_z z$  or  $\sin(k_z z)$ .

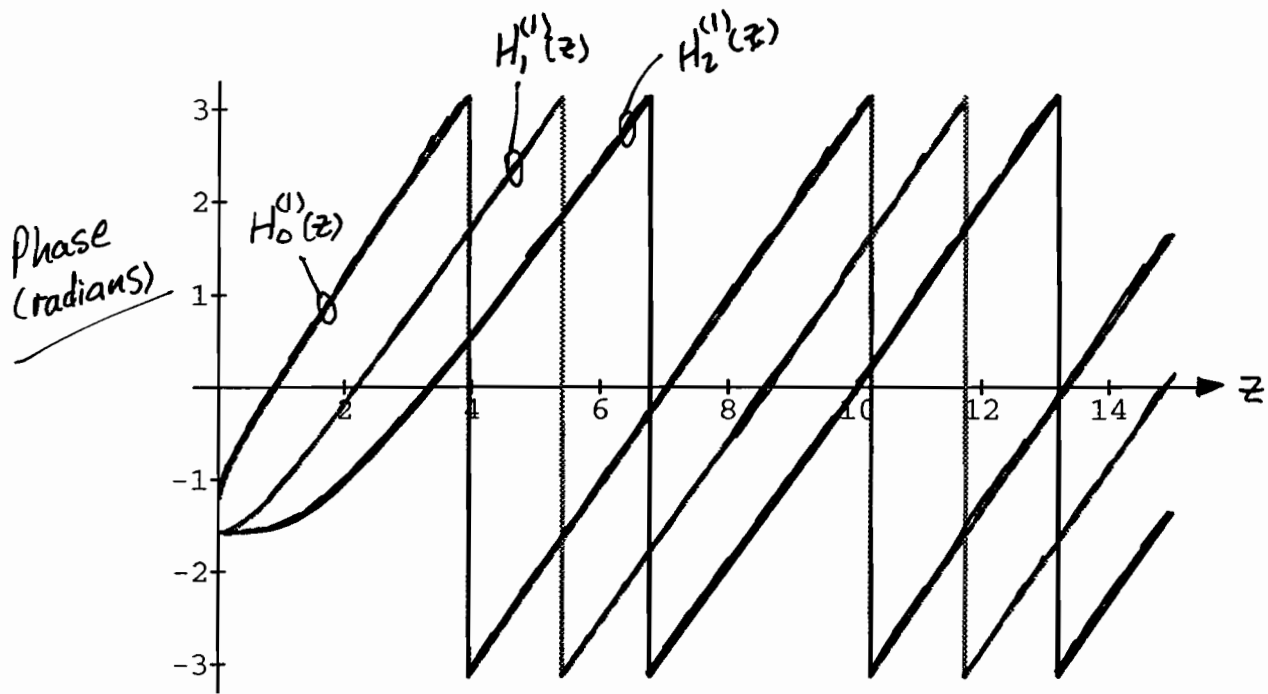
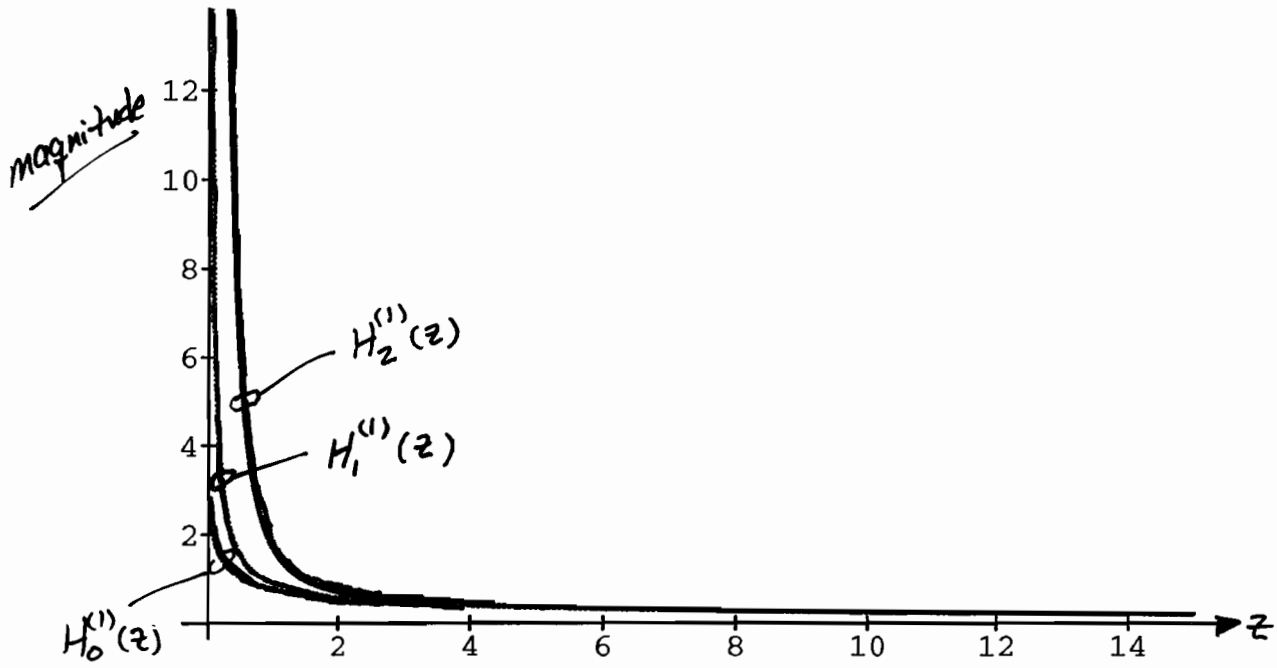
Plots of the various solutions to Bessel's equation are shown on the following pages. Much insight can be gained from the asymptotic forms of these functions-

$$\left. \begin{aligned} J_n(x) &\sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\ N_n(x) &\sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \end{aligned} \right\} \text{as } x \rightarrow \infty$$

$J_n(z)$  and  $N_n(z)$  behave as  
decaying standing waves for  $z > 0$ .

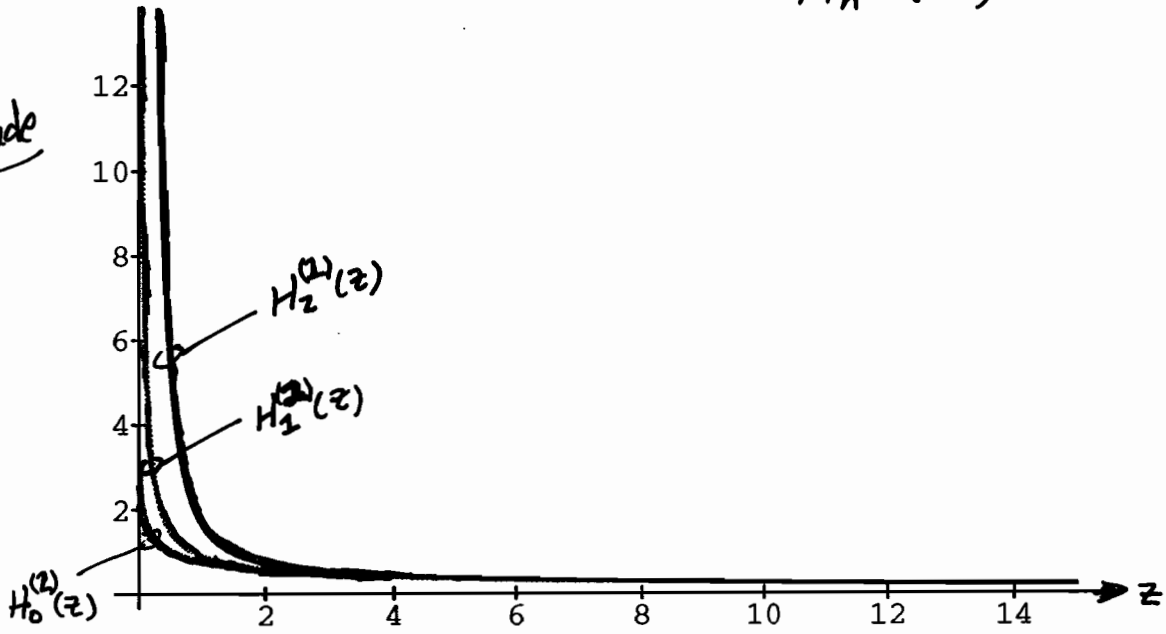


$$H_n^{(1)}(z)$$

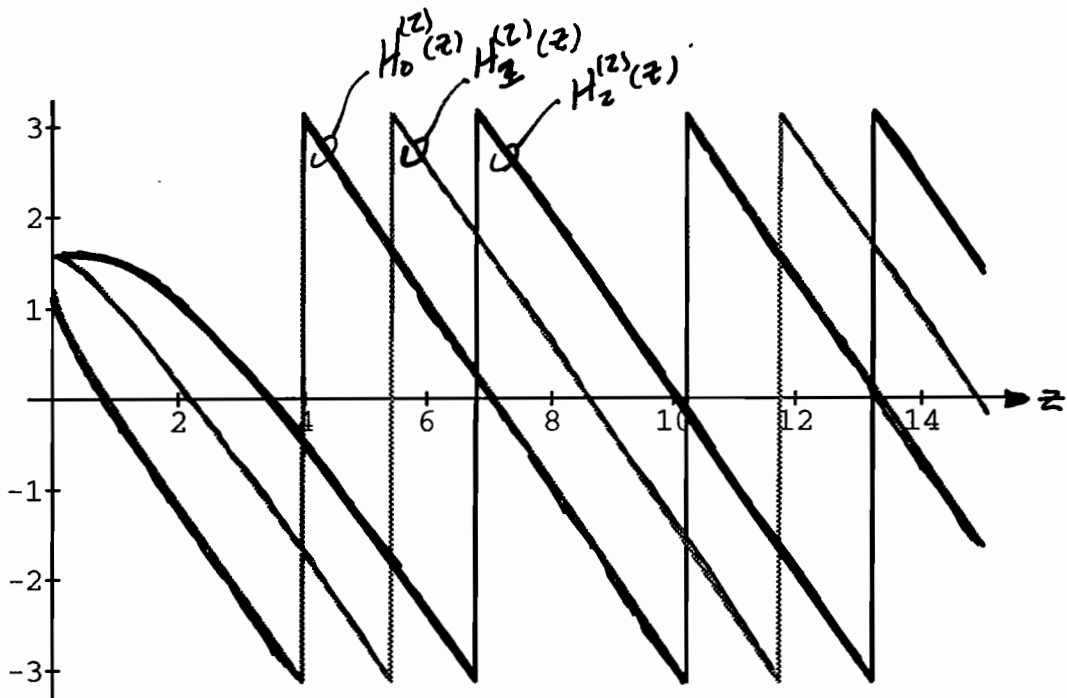


$$H_n^{(z)}(z)$$

Magnitude



Phase (radians)





$$\text{and, } \left. \begin{aligned} H_n^{(1)}(x) &\sim \sqrt{\frac{2}{\pi x}} e^{j(x - \frac{n\pi}{2} - \frac{\pi}{4})} \\ H_n^{(2)}(x) &\sim \sqrt{\frac{2}{\pi x}} e^{-j(x - \frac{n\pi}{2} - \frac{\pi}{4})} \end{aligned} \right\} \text{ as } x \rightarrow \infty$$

- Therefore,
- $J_n(k_p \rho)$  analogous to  $\cos(k_p \rho)$
  - $N_n(k_p \rho)$  analogous to  $\sin(k_p \rho)$
  - $H_n^{(1)}(k_p \rho)$  analogous to  $e^{jk_p \rho}$
  - $H_n^{(2)}(k_p \rho)$  analogous to  $e^{-jk_p \rho}$

for large arguments.

|| As  $x \rightarrow 0$  only  $J_n$  does not become singular.

The functions  $J_n$  and  $N_n$  display the characteristic standing wave behavior, with a  $\frac{1}{\sqrt{\rho}}$  decay for large  $\rho$ . Conversely, the Hankel functions have a characteristic travelling wave behavior (with  $\frac{1}{\sqrt{\rho}}$  decay).

The choice between these two types of solutions depends on the type of problem being solved.

For example, a total solution  $\Psi = f(\rho)g(\phi)h(z)$  may be

$$\Psi = [A J_n(k_p \rho) + B N_n(k_p \rho)] [C e^{jn\phi} + D e^{-jn\phi}] [E e^{jk_z z} + F e^{-jk_z z}]$$

where  $A, B, \dots, F$  are constants. The solution will always be a product, but the specific terms may vary from one problem to another.