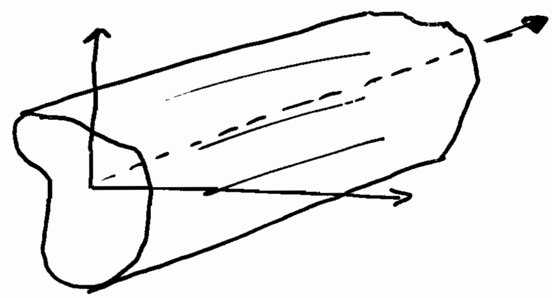
# 2-D Geometries: Solution of the wave equation in Cylindrical coordinates



One type of geometry which receives a great deal of attention in electromagnetics is the 2-D geometry. This means the scattering body is invariant in one dimension. While the scatterer is certainly three dimensional, due to the longitudinal invariance it is reffered to as 2-D.



In order to analyze these problems we must first find the form of the solutions to the wave eqn. in 2-D.

From our discussions earlier, the wave equation for time harmonic fields was found to have the form –

$$\nabla^2 \bar{T} + k^2 \bar{T} = 0 \qquad (1)$$

where $\bar{T} = \bar{E}$ or $\bar{H}$

What we're interested in finding now are the solutions to this equation in cylindrical coordinates –

$$\bar{T} = \hat{\rho}\, T_\rho(\rho,\phi,z) + \hat{\phi}\, T_\phi(\rho,\phi,z) + \hat{z}\, T_z(\rho,\phi,z)$$

Note: (1) cannot be reduced to uncoupled scalar wave equations for $T_\rho, T_\phi$ & $T_z$ since $\nabla^2 \bar{T}$ is defined on the Cartesian components of T.

By definition, $\nabla\times\nabla\times\bar{T} = \nabla(\nabla\cdot\bar{T}) - \nabla^2\bar{T}$

$$\Rightarrow \nabla^2\bar{T} = \nabla(\nabla\cdot\bar{T}) - \nabla\times\nabla\times\bar{T}$$

$\therefore$, the wave eqn ① becomes

$$\nabla(\nabla\cdot\bar{T}) - \nabla\times\nabla\times\bar{T} + k^2\bar{T} = 0 \qquad (2)$$

Expanding out the L.H.S. and noting that

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

where $\psi = \psi(\rho,\phi,z)$ is a scalar function

give <u>three</u> second-order partial differential equations (PDEs):

$$\nabla^2 T_\rho + \left(-\frac{T_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial T_\phi}{\partial\phi}\right) + k^2 T_\rho = 0$$

$$\nabla^2 T_\phi + \left(-\frac{T_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial T_\rho}{\partial\phi}\right) + k^2 T_\phi = 0$$

coupled pde's.
very difficult to solve.

$$\nabla^2 T_z + k^2 T_z = 0 \qquad (3)$$

Since ③ is uncoupled it will be the easiest to work with. Expanding the Laplacian in ③ and letting $\psi = T_z$ gives

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0 \qquad (4)$$

Following the method of <u>Separation of Variables</u>, we will seek solutions of the form

$$\psi = f(\rho)\,g(\phi)\,h(z) \quad \leftarrow \text{product form}$$

Substituting this product form into ④ gives -

$$gh\frac{\partial^2 f}{\partial\rho^2}+\frac{gh}{\rho}\frac{\partial f}{\partial\rho}+\frac{fh}{\rho^2}\frac{\partial^2 g}{\partial\phi^2}+fg\frac{\partial^2 h}{\partial z^2}=-k^2 fgh$$

Dividing by $fgh=\psi$ -

$$\frac{1}{f}\frac{d^2f}{d\rho^2}+\frac{1}{f\rho}\frac{df}{d\rho}+\frac{1}{g\rho^2}\frac{d^2g}{d\phi^2}+\frac{1}{h}\frac{d^2h}{dz^2}+k^2=0 \qquad (5)$$

The last term $\frac{1}{h}\frac{d^2h}{dz^2}$ is <u>not</u> a function of $\rho$ and $\phi$ and must <u>also</u> be independent of $z$ if the L.H.S. of ⑤ is to sum to zero $\forall \rho$ & $\phi$.

$$\Rightarrow \frac{1}{h}\frac{d^2h}{dz^2}=-k_z^2 \quad \text{where } k_z=\text{constant} \qquad (6)$$

Substituting into ⑤ and after multiplication by $\rho^2$ -

$$\frac{\rho^2}{f}\frac{d^2f}{d\rho^2}+\frac{\rho}{f}\frac{df}{d\rho}+\frac{1}{g}\frac{d^2g}{d\phi^2}+\rho^2(k^2-k_z^2)=0 \qquad (7)$$

Now, the $\frac{1}{g}\frac{d^2g}{d\phi^2}$ term is independent of $\rho$ & $z$ and <u>also</u> must be independent of $\phi$ is the L.H.S. of ⑦ is to sum to zero $\forall \rho$ & $z$.

$$\Rightarrow \frac{1}{g}\frac{d^2g}{d\phi^2}=-n^2 \quad \text{where } n=\text{constant} \qquad (8)$$

Substituting ⑧ → ⑦ and multiplying by $f$ gives

$$\rho^2\frac{d^2f}{d\rho^2}+\rho\frac{df}{d\rho}+f\left[-n^2+\rho^2(k^2-k_z^2)\right]=0$$

Defining $k_\rho^2 = k^2 - k_z^2$ gives

$$\rho^2 \frac{d^2 f}{d\rho^2} + \rho \frac{df}{d\rho} + f\left[(k_\rho \rho)^2 - n^2\right] = 0 \qquad (9)$$

This differential equation for $f(\rho)$ is called <u>Bessel's equation of order $n$</u>.

Solutions to (9) are in the form of infinite series — not unlike other tabulated functions like trigonometric functions.

Commonly used solutions to Bessel's equation are -

$f(\rho) =$
- $J_n(k_\rho \rho)$ — Bessel fct., order $n$
- $N_n(k_\rho \rho)$ — Neuman fct., order $n$
- $H_n^{(1)}(k_\rho \rho)$ — Hankel fct. of 1st Kind, order $n$
- $H_n^{(2)}(k_\rho \rho)$ — Hankel fct. of 2nd Kind, order $n$.

Any two of these functions are linearly independent so that $f(\rho)$ is in general a linear combination of any two of them.

From (8) — $\frac{1}{g}\frac{d^2 g}{d\phi^2} = -n^2$ or $\frac{d^2 g}{d\phi^2} + n^2 g = 0$

which has solutions $g(\phi) =$
- $\cos n\phi$
- $\sin n\phi$
- $e^{jn\phi}$
- $e^{-jn\phi}$

Note: in order for single-valued solutions ( $g(\phi) = g(\phi + 2\pi)$ ) implies $n$ = integer.

From ⑥ - $$\frac{1}{h}\frac{d^2h}{dz^2} = -k_z^2 \quad \text{or} \quad \frac{d^2h}{dz^2} + k_z^2 h = 0$$

which has solutions $h(z) =$
- $\cos(k_z z)$
- $\sin(k_z z)$
- $e^{jk_z z}$
- $e^{-jk_z z}$

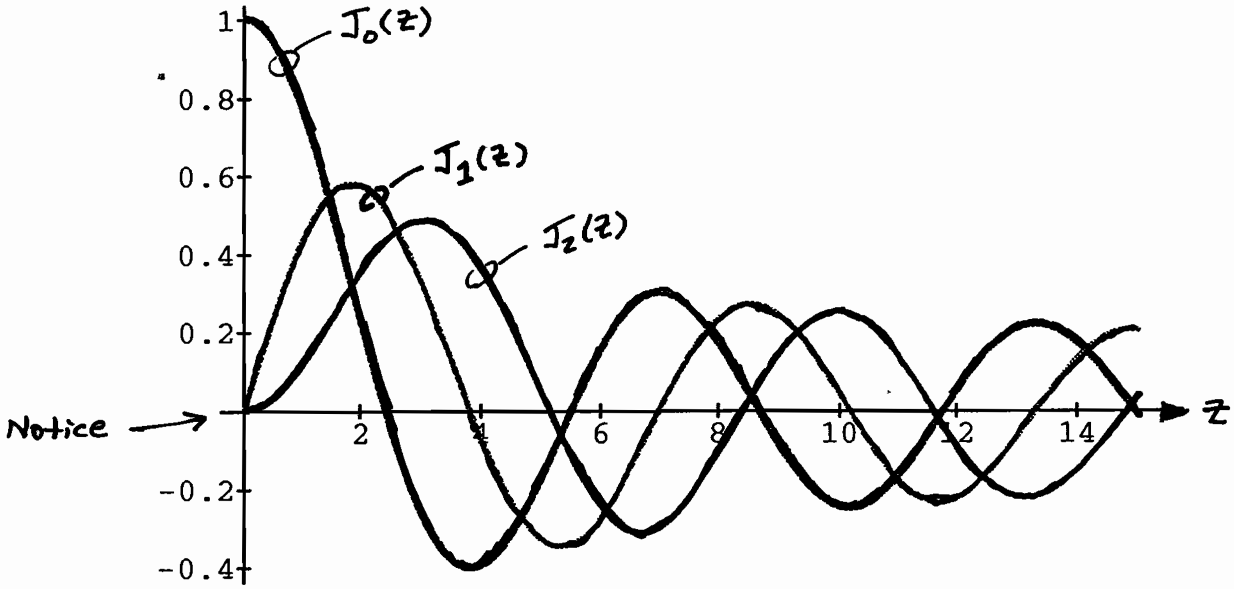
The specific linear combinations used for $f(\rho)$, $g(\phi)$ and $h(z)$ depend a great deal on the type of problem which is to be solved.

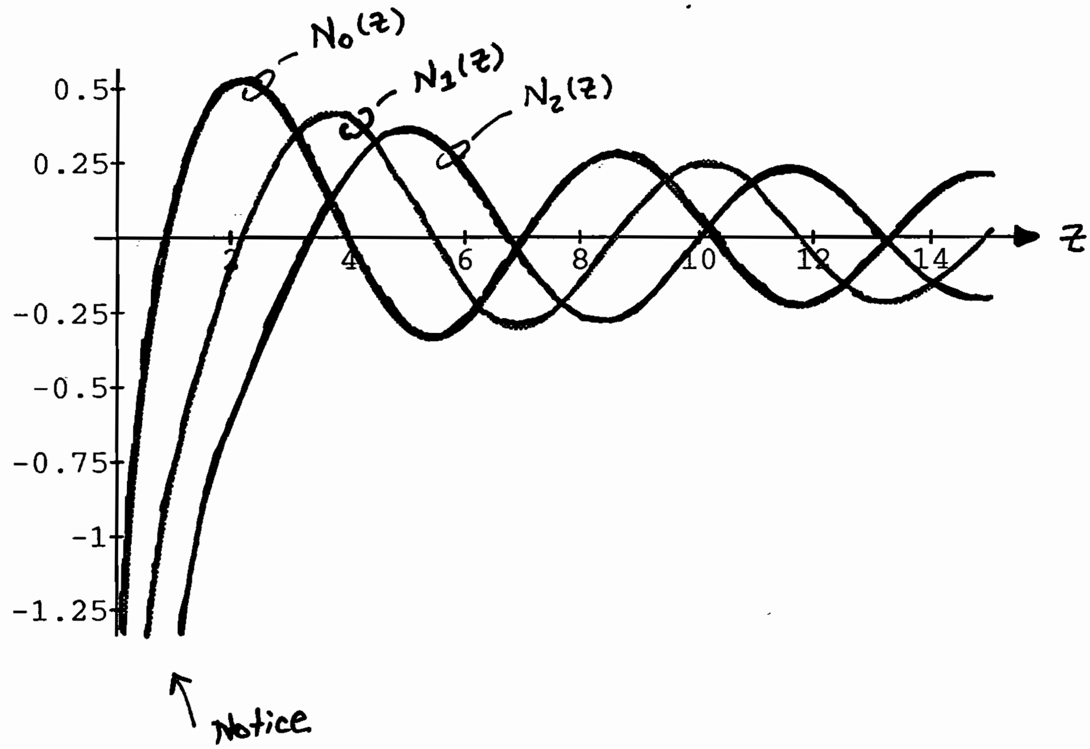
For example, in a cylindrical waveguide, propagating (or evanescent) modes would be adequately represented by an $e^{\pm jk_z z}$ dependence. In a resonant cavity would likely used $\cos k_z z$ or $\sin(k_z z)$.

Plots of the various solutions to Bessel's equation are shown on the following pages. Much insight can be gained from the asymptotic forms of these functions-

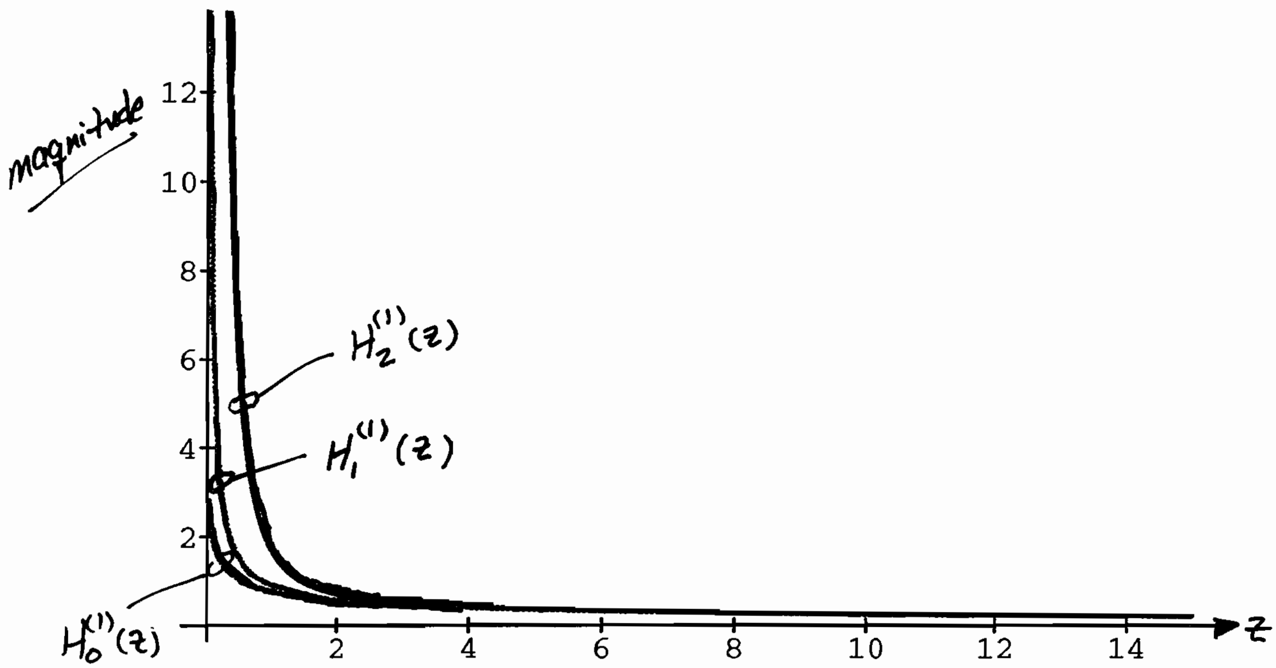
$$\left.\begin{aligned} J_n(x) &\sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\ N_n(x) &\sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \end{aligned}\right\} \text{as } x \to \infty$$

$J_n(z)$ and $N_n(z)$ behave as decaying standing waves for $z>0$.

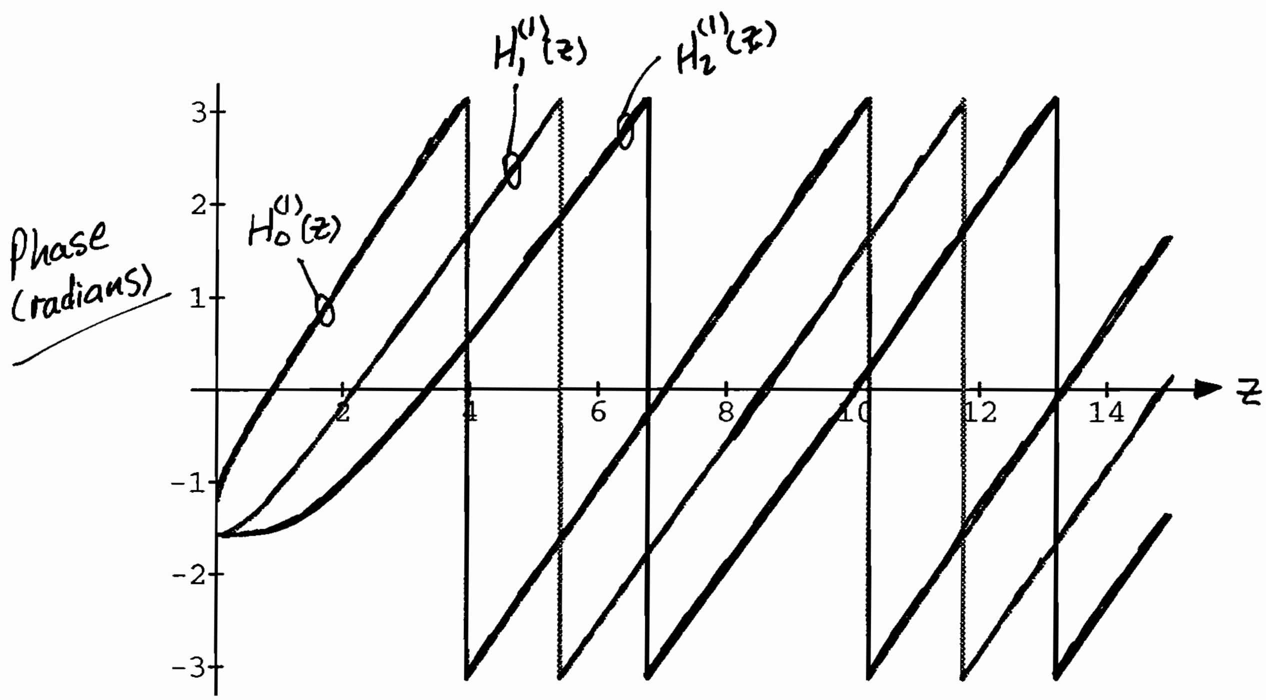


$H_n^{(1)}(z)$

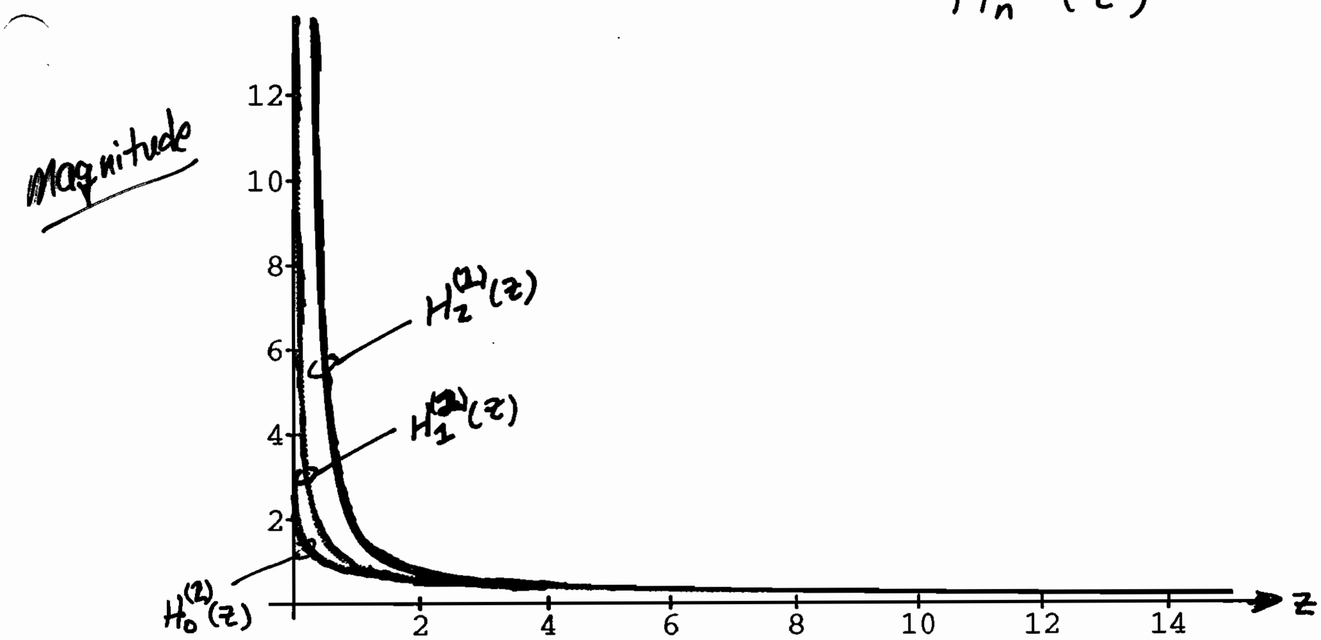


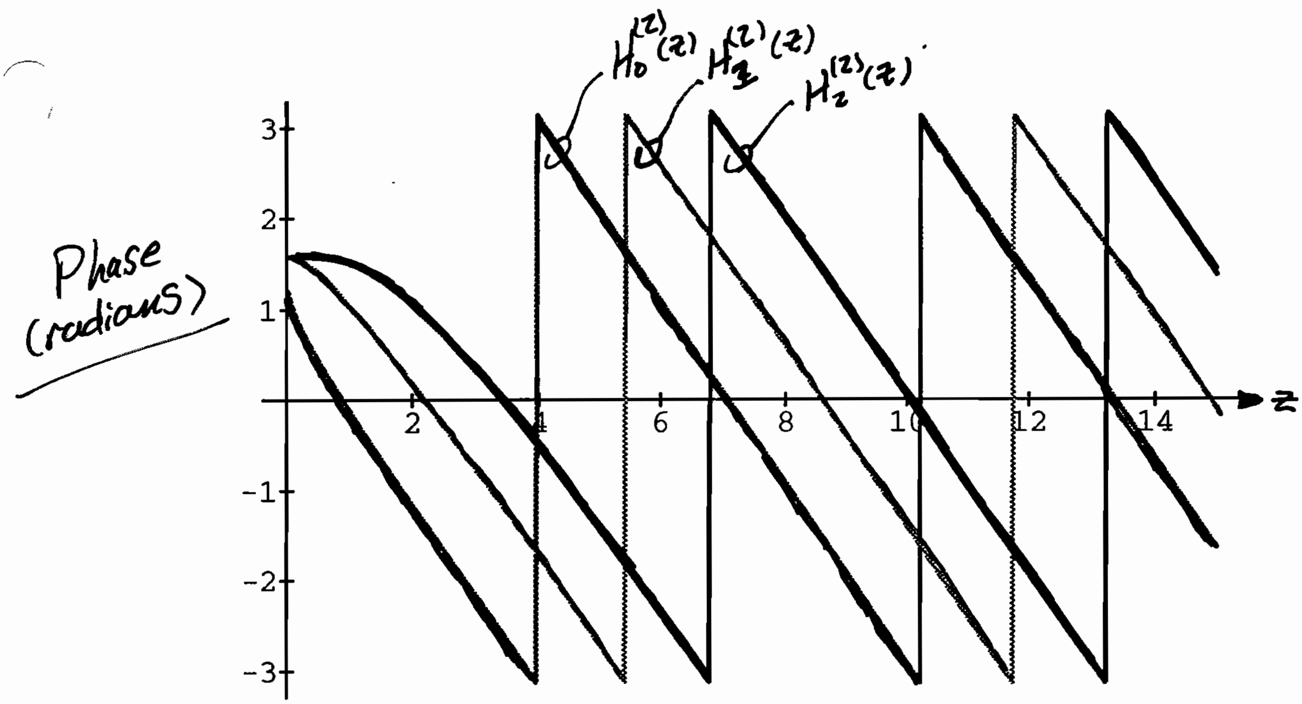
magnitude
12
10
8
6
4
2
2
4
6
8
10
12
14
z
$H_2^{(1)}(z)$
$H_1^{(1)}(z)$
$H_0^{(1)}(z)$

Phase (radians)
3
2
1
-1
-2
-3
2
4
6
8
10
12
14
z
$H_0^{(1)}(z)$
$H_1^{(1)}(z)$
$H_2^{(1)}(z)$

$H_n^{(2)}(z)$


Magnitude
12
10
8
6
4
2
$H_2^{(2)}(z)$
$H_1^{(2)}(z)$
$H_0^{(2)}(z)$
2
4
6
8
10
12
14
z


$H_0^{(2)}(z)$
$H_1^{(2)}(z)$
$H_2^{(2)}(z)$
Phase (radians)
3
2
1
-1
-2
-3
2
4
6
8
10
12
14
z

and,
$$\left.\begin{aligned} H_n^{(1)}(x) &\sim \sqrt{\frac{2}{\pi x}}\, e^{j(x-\frac{n\pi}{2}-\frac{\pi}{4})} \\ H_n^{(2)}(x) &\sim \sqrt{\frac{2}{\pi x}}\, e^{-j(x-\frac{n\pi}{2}-\frac{\pi}{4})} \end{aligned}\right\} \text{ as } x \to \infty$$

Therefore,
- $J_n(k_\rho \rho)$ analogous to $\cos(k_\rho \rho)$
- $N_n(k_\rho \rho)$ analogous to $\sin(k_\rho \rho)$

- $H_n^{(1)}(k_\rho \rho)$ analogous to $e^{jk_\rho \rho}$
- $H_n^{(2)}(k_\rho \rho)$ analogous to $e^{-jk_\rho \rho}$

for large arguments.

|| As $x \to 0$ only $J_n$ does not become singular.

The functions $J_n$ and $N_n$ display the characteristic standing wave behavior, with a $\frac{1}{\sqrt{\rho}}$ decay for large $\rho$. Conversely, the Hankel functions have a characteristic travelling wave behavior (with $1/\sqrt{\rho}$ decay).

The choice between these two types of solutions depends on the type of problem being solved.

For example, a total solution $\psi = f(\rho)g(\phi)h(z)$ may be

$$\psi = \left[A J_n(k_\rho \rho) + B N_n(k_\rho \rho)\right]\left[C e^{jn\phi} + D e^{-jn\phi}\right]\left[E e^{jk_z z} + F e^{-jk_z z}\right]$$

where $A, B, \cdots, F$ are constants. The solution will always be a product, but the specific terms may vary from one problem to another.