

It is possible to transport energy from one position to another using EM fields. No surprise: we experience this everyday through cell phones, radios, sat TV.

Poynting's theorem is the mathematical definition for the conservation of power in EM fields. With reference to a volume V surrounded by S , Poynting's theorem is

$$P_s = P_e + P_d + \frac{d}{dt} (W_e + W_m) \quad (1-59a), (1)$$

where $\bullet P_s = \text{EM power supplied to } V = - \int_{V(S)} (\vec{E} \cdot \vec{J}_i + \vec{H} \cdot \vec{M}_i) dV \quad [W] \quad (1-59a), (2)$

$\bullet P_e = \text{power leaving } V \text{ through } S = \oint_{S(V)} (\vec{E} \times \vec{H}) \cdot d\vec{S} \quad [W] \quad (1-59b), (3)$

$\bullet P_d = \text{power dissipated in } V = \int_{V(S)} (\vec{E} \cdot \vec{J}_c) dV \quad [W] \quad (1-59c), (4)$

$\bullet W_e = \text{Electric energy stored in } V = \int_{V(S)} \frac{1}{2} \epsilon |\vec{E}|^2 dV \quad [J] \quad (1-59d), (5)$

$\bullet W_m = \text{Magnetic energy stored in } V = \int_{V(S)} \frac{1}{2} \mu |\vec{H}|^2 dV \quad [J] \quad (1-59e), (6)$

In words, Poynting's theorem (1) states that the EM power supplied to a volume equals the sum of (1) the EM power leaving the volume, (2) the power dissipated in V (through an irreversible process), and (3) the time rate of change in the EM energy stored in V .

The quantity $\vec{S} = \vec{E} \times \vec{H} \quad [W/m^2]$ is the Poynting vector, widely interpreted as the ^{EM} power density vector. This vector (often) indicates power flow, though there may be ambiguity in rare circumstances.

The correctness of (3) is not in doubt, as the total power leaving S . It is the interpretation of \vec{S} as the intensity of energy flow at a point that may fail. *

* See J.A. Stratton, "Electromagnetic Theory,"
McGraw-Hill, 1941, pp. 131-137.

Indeed, we can add any vector field $\vec{\mathcal{P}}'$ to $\vec{\mathcal{P}}$
s.t. $\oint_{S(V)} \vec{\mathcal{P}}' \cdot d\vec{S} = 0$ w/o changing (3). For example,

if a vector field $\vec{\mathcal{P}}'$ is solenoidal throughout V , \Rightarrow
non-zero

$$\nabla \cdot \vec{\mathcal{P}}'(F) = 0 \quad \forall (F) \in \{V\} \quad (7)$$

Then, applying the divergence theorem

$$\int_{V(S)} [\nabla \cdot \vec{\mathcal{P}}'(F)] dV = \oint_{S(V)} \vec{\mathcal{P}}'(F) \cdot d\vec{S} \quad (8)$$

Using (7) in (8) gives $\oint_{S(V)} \vec{\mathcal{P}}' \cdot d\vec{S} = 0$ (9)

Therefore, the Poynting vector at every point in V
is $\vec{\mathcal{P}}(F) + \vec{\mathcal{P}}'(F)$, but the total power leaving S
is $\oint_{S(V)} \vec{\mathcal{P}}(F) \cdot d\vec{S}$.

The vast majority of our work this semester will be with sinusoidally steady state solutions to ME's. We will assume an $e^{j\omega t}$ time dependence, & work w/ vector phasor fields that have only spatial dependence.

For example, w/ $\bar{\mathbf{E}}(\mathbf{r}, t) = \text{Re}\{\bar{\mathbf{E}}(\mathbf{r})e^{j\omega t}\}$ (1-61a), (10)

and $\bar{\mathbf{B}}(\mathbf{r}, t) = \text{Re}\{\bar{\mathbf{B}}(\mathbf{r})e^{j\omega t}\}$ (1-61d), (11)

Then the point form of Faraday's law becomes

$$\nabla \times \text{Re}\{\bar{\mathbf{E}}(\mathbf{r})e^{j\omega t}\} = -\frac{d}{dt} \text{Re}\{\bar{\mathbf{B}}(\mathbf{r})e^{j\omega t}\}$$

The operator $\text{Re}\{\}$ & differentiation commute. This gives

$$\text{Re}\{\nabla \times [\bar{\mathbf{E}}(\mathbf{r})e^{j\omega t}]\} = -\text{Re}\left\{\frac{d}{dt}[\bar{\mathbf{B}}(\mathbf{r})e^{j\omega t}]\right\}$$

or

$$\text{Re}\{[\nabla \times \bar{\mathbf{E}}(\mathbf{r})]e^{j\omega t}\} = -\text{Re}\left\{\bar{\mathbf{B}}(\mathbf{r})\frac{d}{dt}(e^{j\omega t})\right\}$$

Therefore, $\text{Re}\{[\nabla \times \bar{\mathbf{E}}(\mathbf{r})]e^{j\omega t}\} = \text{Re}\{[-j\omega \bar{\mathbf{B}}(\mathbf{r})]e^{j\omega t}\}$ (12)

It can be shown that if $A \doteq B$ are complex quantities and $\text{Re}\{Ae^{j\omega t}\} = \text{Re}\{Be^{j\omega t}\}$ for all t , then $A=B$.

Applying this lemma to (12) gives the phasor form of Faraday's law.

$$\nabla \times \bar{\mathbf{E}}(\mathbf{r}) = -j\omega \bar{\mathbf{B}}(\mathbf{r}) \quad (13)$$

Table 1-4 in the text lists the phasor form of ME'S in both point & integral form.

TABLE 1-4
Instantaneous and time-harmonic forms of Maxwell's equations and continuity equation in differential and integral forms

Instantaneous	Time harmonic
Differential form	
$\nabla \times \mathcal{E} = -\mathcal{M}_i - \frac{\partial \mathcal{D}}{\partial t}$	$\nabla \times \mathbf{E} = -\mathbf{M}_i - j\omega \mathbf{B}$
$\nabla \times \mathcal{H} = \mathcal{J}_i + \mathcal{J}_c + \frac{\partial \mathcal{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J}_i + \mathbf{J}_c + j\omega \mathbf{D}$
$\nabla \cdot \mathcal{D} = q_{ev}$	$\nabla \cdot \mathbf{D} = q_{ev}$
$\nabla \cdot \mathcal{H} = q_{mv}$	$\nabla \cdot \mathbf{B} = q_{mv}$
$\nabla \cdot \mathcal{J}_{ic} = -\frac{\partial q_{ev}}{\partial t}$	$\nabla \cdot \mathbf{J}_{ic} = -j\omega q_{ev}$
Integral form	
$\oint_C \mathcal{E} \cdot d\mathbf{l} = -\iint_S \mathcal{M}_i \cdot d\mathbf{s} - \frac{\partial}{\partial t} \iint_S \mathcal{D} \cdot d\mathbf{s}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\iint_S \mathbf{M}_i \cdot d\mathbf{s} - j\omega \iint_S \mathbf{B} \cdot d\mathbf{s}$
$\oint_C \mathcal{H} \cdot d\mathbf{l} = \iint_S \mathcal{J}_i \cdot d\mathbf{s} + \iint_S \mathcal{J}_c \cdot d\mathbf{s} + \frac{\partial}{\partial t} \iint_S \mathcal{D} \cdot d\mathbf{s}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J}_i \cdot d\mathbf{s} + \iint_S \mathbf{J}_c \cdot d\mathbf{s} + j\omega \iint_S \mathbf{D} \cdot d\mathbf{s}$
$\oiint_S \mathcal{D} \cdot d\mathbf{s} = Q_e$	$\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q_e$
$\oiint_S \mathcal{H} \cdot d\mathbf{s} = Q_m$	$\oiint_S \mathbf{B} \cdot d\mathbf{s} = Q_m$
$\oiint_S \mathcal{J}_{ic} \cdot d\mathbf{s} = -\frac{\partial Q_e}{\partial t}$	$\oiint_S \mathbf{J}_{ic} \cdot d\mathbf{s} = -j\omega Q_e$

In electrical engineering, the time dependence $e^{j\omega t}$ is very commonly used. However, in physics it is common to use the dependence $e^{-i\omega t}$. Faraday's law can then written as

$$\nabla \times \vec{E}(F) = i\omega \vec{B}(F)$$

To convert results in a physics text or paper w/ this convention, make the substitution $i \rightarrow -j$.

Occasionally you may come across a paper using $e^{-j\omega t}$ or $e^{i\omega t}$ convention. Need to be careful when using results from unfamiliar authors.