Whites
EE692

Lecture 2 – Power & Energy Time-harmonic EM Fields

It is possible to transport energy from one position to another using EM fields. No surprise: we experience this everyday through cell phones, radios, sat TV.

Poynting's Theorem is the mathematical definition for the conservation of power in EM fields. With reference to a volume V surrounded by S, Poynting's theorem is

$$\mathcal{P}_s = \mathcal{P}_e + \mathcal{P}_d + \frac{d}{dt}\left(\mathcal{W}_e + \mathcal{W}_m\right) \qquad (1\text{-}60a), (1)$$

where
- $\mathcal{P}_s$ = EM power supplied to V $= -\int_{V(S)} \left(\bar{\mathcal{E}}\cdot\bar{\mathcal{J}}_i + \bar{\mathcal{H}}\cdot\bar{\mathcal{M}}_i\right) dv$ [W] (1-59a), (2)
- $\mathcal{P}_e$ = power leaving V through S $= \oint_{S(V)} \left(\bar{\mathcal{E}}\times\bar{\mathcal{H}}\right)\cdot d\bar{s}$ [W] (1-57), (3)
- $\mathcal{P}_d$ = power dissipated in V $= \int_{V(S)} \left(\bar{\mathcal{E}}\cdot\bar{\mathcal{J}}_c\right) dv$ [W] (1-59c), (4)
- $\mathcal{W}_e$ = Electric energy stored in V $= \int_{V(S)} \frac{1}{2}\epsilon\left|\bar{\mathcal{E}}\right|^2 dv$ [J] (1-59d), (5)
- $\mathcal{W}_m$ = Magnetic energy stored in V $= \int_{V(S)} \frac{1}{2}\mu\left|\bar{\mathcal{H}}\right|^2 dv$ [J] (1-59b), (6)

In words, Poynting's theorem (1) states that the EM power supplied to a volume equals the sum of (1) the EM power leaving the volume, (2) the power dissipated in V (through an irreversible process), and (3) the time rate of change in the EM energy stored in V.

The quantity $\bar{\mathcal{S}} = \bar{\mathcal{E}}\times\bar{\mathcal{H}}$ [W/m²] is the Poynting vector widely interpreted as the EM power density vector. This vector (often) indicates power flow, though there may ambiguity in rare circumstances.

The correctness of (3) is not in doubt, as the <u>total</u> power leaving S. It is the interpretation of $\bar{\mathcal{S}}$ as the intensity of energy flow <u>at a point</u> that may fail.*

* See J.A. Stratton, "Electromagnetic Theory," McGraw-Hill, 1941, pp. 131–137.

Indeed, we can add any vector field $\bar{\mathcal{S}}'$ to $\bar{\mathcal{S}}$ s.t. $\oint_{S(V)} \bar{\mathcal{S}}' \cdot d\bar{s} = 0$ w/o changing (3). For example, if a non-zero vector field $\bar{\mathcal{S}}'$ is solenoidal throughout V, $\Rightarrow$

$$\nabla \cdot \bar{\mathcal{S}}'(\bar{r}) = 0 \qquad \forall (\bar{r}) \in \{V\} \tag{7}$$

Then, applying the divergence theorem

$$\int_{V(S)} \left[\nabla \cdot \bar{\mathcal{S}}'(\bar{r})\right] dv = \oint_{S(V)} \bar{\mathcal{S}}'(\bar{r}) \cdot d\bar{s} \tag{8}$$

Using (7) in (8) gives $\oint_{S(V)} \bar{\mathcal{S}}' \cdot d\bar{s} = 0$ (9)

Therefore, the Poynting vector at every point in V is $\bar{\mathcal{S}}(\bar{r}) + \bar{\mathcal{S}}'(\bar{r})$, but the total power leaving S is

$$\oint_{S(V)} \bar{\mathcal{S}}(\bar{r}) \cdot d\bar{s} .$$

Time-Harmonic EM Fields

The vast majority of our work this semester will be with sinusoidally steady state solutions to ME's. We will assume an $e^{j\omega t}$ time dependence, & work w/ vector phasor fields that have only spatial dependence.

For example, w/ $\bar{\mathcal{E}}(\bar{r},t) = Re\{\bar{E}(\bar{r})e^{j\omega t}\}$ (1-61a), (10)

and $\bar{\mathcal{B}}(\bar{r},t) = Re\{\bar{B}(\bar{r})e^{j\omega t}\}$ (1-61d), (11)

then the point form of Faraday's law becomes

$$\nabla \times Re\{\bar{E}(\bar{r})e^{j\omega t}\} = -\frac{d}{dt} Re\{\bar{B}(\bar{r})e^{j\omega t}\}$$

The operator $Re\{\}$ & differentiation commute. This gives

$$Re\left\{\nabla \times \left[\bar{E}(\bar{r})e^{j\omega t}\right]\right\} = -Re\left\{\frac{d}{dt}\left[\bar{B}(\bar{r})e^{j\omega t}\right]\right\}$$

or

$$Re\left\{\left[\nabla \times \bar{E}(\bar{r})\right]e^{j\omega t}\right\} = -Re\left\{\bar{B}(\bar{r})\frac{d}{dt}\left(e^{j\omega t}\right)\right\}$$

Therefore,

$$Re\left\{\left[\nabla \times \bar{E}(\bar{r})\right]e^{j\omega t}\right\} = Re\left\{\left[-j\omega\bar{B}(\bar{r})\right]e^{j\omega t}\right\} \quad (12)$$

It can be shown that if A & B are complex quantities and $Re\{Ae^{j\omega t}\} = Re\{Be^{j\omega t}\}$ for all t, then $A = B$.

Applying this lemma to (12) gives the phasor form of Faraday's law.

$$\nabla \times \bar{E}(\bar{r}) = -j\omega\bar{B}(\bar{r}) \quad (13)$$

Table 1-4 in the text lists the phasor form of ME's in both point & integral form.

TABLE 1-4
Instantaneous and time-harmonic forms of Maxwell's equations and continuity equation in differential and integral forms

Instantaneous	Time harmonic
Differential form	
$\nabla \times \mathscr{E} = -\mathscr{M}_i - \dfrac{\partial \mathscr{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\mathbf{M}_i - j\omega \mathbf{B}$
$\nabla \times \mathscr{H} = \mathscr{J}_i + \mathscr{J}_c + \dfrac{\partial \mathscr{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J}_i + \mathbf{J}_c + j\omega \mathbf{D}$
$\nabla \cdot \mathscr{D} = \mathscr{q}_{ev}$	$\nabla \cdot \mathbf{D} = q_{ev}$
$\nabla \cdot \mathscr{B} = \mathscr{q}_{mv}$	$\nabla \cdot \mathbf{B} = q_{mv}$
$\nabla \cdot \mathscr{J}_{ic} = -\dfrac{\partial \mathscr{q}_{ev}}{\partial t}$	$\nabla \cdot \mathbf{J}_{ic} = -j\omega q_{ev}$
Integral form	
$\oint_C \mathscr{E} \cdot d\mathbf{l} = -\iint_S \mathscr{M}_i \cdot d\mathbf{s} - \dfrac{\partial}{\partial t}\iint_S \mathscr{B} \cdot d\mathbf{s}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\iint_S \mathbf{M}_i \cdot d\mathbf{s} - j\omega \iint_S \mathbf{B} \cdot d\mathbf{s}$
$\oint_C \mathscr{H} \cdot d\mathbf{l} = \iint_S \mathscr{J}_i \cdot d\mathbf{s} + \iint_S \mathscr{J}_c \cdot d\mathbf{s} + \dfrac{\partial}{\partial t}\iint_S \mathscr{D} \cdot d\mathbf{s}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J}_i \cdot d\mathbf{s} + \iint_S \mathbf{J}_c \cdot d\mathbf{s} + j\omega \iint_S \mathbf{D} \cdot d\mathbf{s}$
$\oiint_S \mathscr{D} \cdot d\mathbf{s} = \mathscr{Q}_e$	$\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q_e$
$\oiint_S \mathscr{B} \cdot d\mathbf{s} = \mathscr{Q}_m$	$\oiint_S \mathbf{B} \cdot d\mathbf{s} = Q_m$
$\oiint_S \mathscr{J}_{ic} \cdot d\mathbf{s} = -\dfrac{\partial \mathscr{Q}_e}{\partial t}$	$\oiint_S \mathbf{J}_{ic} \cdot d\mathbf{s} = -j\omega Q_e$

In electrical engineering, the time dependence $e^{j\omega t}$ is very commonly used. However, in physics it is common to use the dependence $e^{-i\omega t}$. Faraday's law can then written as

$$\nabla \times \bar{E}(\bar{r}) = i\omega \bar{B}(\bar{r})$$

To convert results in a physics text or paper w/ this convention, make the substitution $i \Rightarrow -j$.

Occasionally you make come a across a paper using $e^{-j\omega t}$ or $e^{i\omega t}$ convention. Need to be careful when using results from unfamiliar authors.