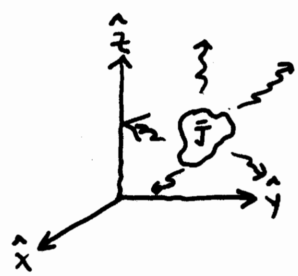
Source/field Relationships

We will shortly require, in our numerical solution methods, the ability to find the fields produced by a given distribution of electric current density: source/field relationships.

Assuming the source is varying in a time-harmonic fashion with $e^{j\omega t}$, then Maxwell's equations may be written as



$$\nabla \times \bar{E} = -j\omega\mu \bar{H} \qquad (1)$$
$$\nabla \times \bar{H} = j\omega\epsilon \bar{E} + \bar{J} \qquad (2)$$

The objective is to solve for the fields produced by this current distribution $\bar{J}$. The solutions, as we shall see, may be expressed as a convolution integral.

Method 1: The Potential Method

Taking $\nabla \cdot (1) \Rightarrow \quad \nabla \cdot \nabla \times \bar{E} = -j\omega\mu \nabla \cdot \bar{H}$ (homogeneous)

or $\quad 0 = \nabla \cdot \bar{H}$

This type of field is called solenoidal.

A solenoidal field may be expressed as the curl of another vector (Helmholtz thm) giving $$\bar{H} = \nabla \times \bar{A} \qquad (3)$$

Sub. this result into (1) gives $\nabla \times \bar{E} = -j\omega\mu \nabla \times \bar{A}$

or, $$\nabla \times [\underbrace{\bar{E} + j\omega\mu \bar{A}}] = 0$$

A vector field which is curl-free is called <u>irrotational</u> or <u>conservative</u> & can be expressed as the gradient of another scalar potential →

$$\bar{E} + j\omega\mu \bar{A} = \underset{\substack{\uparrow \\ \text{by convention}}}{-}\nabla \Phi_e \qquad (4)$$

Now, sub. both (4) & (3) into (2) giving -

$$\nabla \times \nabla \times \bar{A} = j\omega\epsilon[-j\omega\mu \bar{A} - \nabla \Phi_e] + \bar{J}$$

vector i.d. $\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$

[where $\nabla^2 \bar{A}$ is defined on the <u>cartesian</u> components of $\bar{A}$ only! ($\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z$)]

∴

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} - \omega^2\mu\epsilon \bar{A} + j\omega\epsilon\nabla\Phi_e = \bar{J}$$

rewriting - $$\nabla^2 \bar{A} + k^2 \bar{A} - \nabla(\nabla \cdot \bar{A} + j\omega\epsilon \Phi_e) = -\bar{J}$$

By the Helmholtz theorem, a vector field is uniquely specified (within an additive constant) when both the curl and divergence are specified.

- We've selected $\nabla \times \bar{A} = \bar{H}$
- Now choose $\nabla \cdot \bar{A} = -j\omega\epsilon\,\Phi_e$ to simplify the last eqn.

This choice is called the Lorentz Gauge. For time-varying fields it is consistent w/ conservation of charge. [$\nabla \cdot \bar{A} = 0 \Rightarrow$ Coulomb Gauge.]

Substituting this last choice for $\nabla \cdot \bar{A}$ gives -

$$\underline{\underline{\nabla^2 \bar{A} + k^2 \bar{A} = -\bar{J}}} \qquad (5)$$

A vector Helmholtz eqn. for $\bar{A}$. It is a set of 3 scalar equations - inhomogeneous O.D.E.'s

Once this equation has been solved for $\bar{A}$, then the fields are given as -

- $\bar{H} = \nabla \times \bar{A}$

& $\bar{E} = -j\omega\mu\bar{A} - \nabla\Phi_e$ or w/ $\Phi_e = -\frac{1}{j\omega\epsilon}\nabla \cdot \bar{A}$

$\Rightarrow$ • $\bar{E} = -j\omega\mu\bar{A} + \frac{1}{j\omega\epsilon}\nabla\nabla \cdot \bar{A}$

Method 2 : The Direct Method

$$\nabla \times \bar{E} = -j\omega\mu \bar{H} \quad (1) \quad ; \quad \nabla \times \bar{H} = j\omega\epsilon \bar{E} + \bar{J} \quad (2)$$

Take $\nabla \times (1) \rightarrow \quad \nabla \times \nabla \times \bar{E} = -j\omega\mu \nabla \times \bar{H}$

Sub (2) into this giving - $\quad \nabla \times \nabla \times \bar{E} = -j\omega\mu (j\omega\epsilon \bar{E} + \bar{J})$

or)
$$\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} - \omega^2\mu\epsilon \bar{E} = -j\omega\mu \bar{J} \qquad (6)$$

But what is $\nabla \cdot \bar{E}$? Take $(\nabla \cdot)$ of (2)

$$\nabla \cdot \nabla \times \bar{H} = 0 = j\omega\epsilon \nabla \cdot \bar{E} + \nabla \cdot \bar{J}$$

or)
$$\nabla \cdot \bar{E} = -\frac{1}{j\omega\epsilon} \nabla \cdot \bar{J}$$

Sub. into (6) gives

$$-\frac{1}{j\omega\epsilon} \nabla\nabla \cdot \bar{J} - \nabla^2 \bar{E} - k^2 \bar{E} = -j\omega\mu \bar{J}$$

or)
$$\underline{\underline{\nabla^2 \bar{E} + k^2 \bar{E} = j\omega\mu \bar{J} - \frac{1}{j\omega\epsilon} \nabla\nabla \cdot \bar{J}}} \qquad (7)$$

Once $\bar{E}$ has been solved, can find $\bar{H}$ from Maxwell's equations.

In many other EM courses, primarily undergraduate, this is about the extent of the source/field concept.

Typically at this point, the current density $\bar{J}$ would be assumed to have some functional form from which the fields could mathematically be computed using the potential Method (5) & Maxwell's eqns.

However, in this course we will be <u>solving</u> for either ρ_e or $\bar{J}$! They are the unknowns.

Once they have been found (either ρ_e or $\bar{J}$, depending on the problem), we can calculate all other fields, power, etc...

<u>Problem</u> - From (5) & (7) we don't have expessions of the form $\bar{A} = (\cdots)$ or $\bar{E} = (\cdots)$.

Instead we have $(\nabla^2 + k^2)\bar{A} = -\bar{J}$

& $(\nabla^2 + k^2)\bar{E} = -\dfrac{\nabla\nabla\cdot\bar{J} + k^2\bar{J}}{j\omega\epsilon}$

<u>Form</u> of L.H.S is the same

<u>Forcing</u> fcts. are different.

(*) To actually solve for $\bar{A}$ or $\bar{E}$ we must <u>invert</u> this linear operator $\nabla^2 + k^2$.