

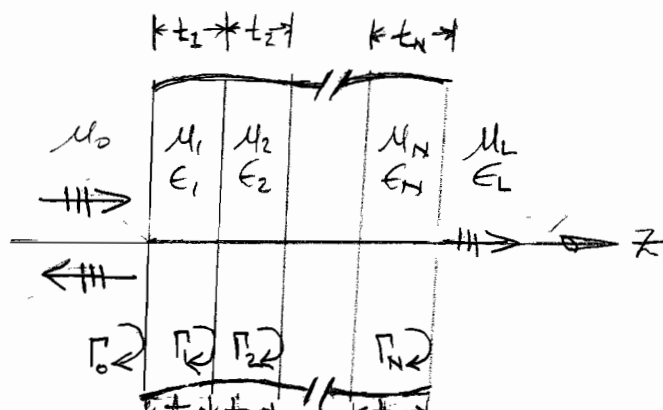
We will now apply some of the principles we've learned in the previous two lectures to the design of multi-layered slab structures with the intention of producing a small reflection over a desired frequency bandwidth.

There are a number of approaches. The most accurate would be to use the exact slab scattering equations; but these would quickly become too complicated for an analytical solution. Another approach would be to use these exact equations with a numerical minimization algorithm such as genetic algorithms (GA), simulated annealing (SA), or another.

Could also use TMs in ADS & the built-in optimization.

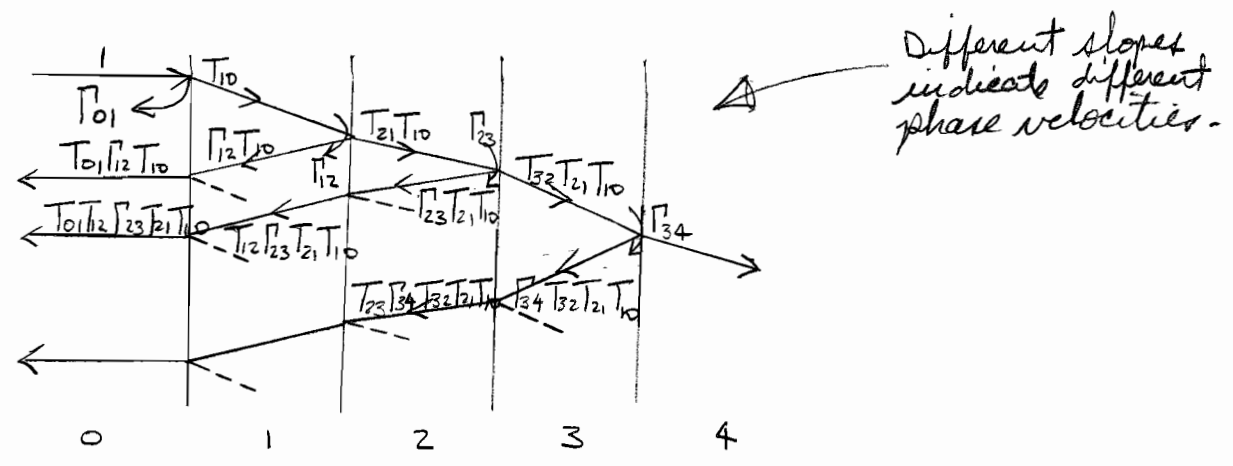
Here, we will use an approximate approach that works quite well when the reflections are small, which is what we want.

The geometry is that shown in Fig 5-20 of the text:



Our task is to choose the # of layers, their material parameters, & their thicknesses to provide a "small" reflection at the front face. That is, we wish to design a multi-section impedance transformer (See Pozar, Sect 5.5).

To a first approximation, the ref. coeff at the front face is due to the primary <sup>wave</sup> reflections from each of the interfaces. A ladder diagram is helpful:



If the reflections are small, then to a first-order approx:

$$\Gamma_{in} \approx \Gamma_{01} + T_{01} \Gamma_{12} T_{10} e^{-j2\phi_1} + T_{01} T_{12} \Gamma_{23} T_{21} T_{10} e^{-j2(\phi_1 + \phi_2)} + \dots \quad (1)$$

Since the reflections are small, then these partial transmission coeff will be  $\approx 1$ . Consequently,

$$\Gamma_{in} \approx \Gamma_{01} + \Gamma_1 e^{-j2\phi_1} + \Gamma_2 e^{-j2(\phi_1 + \phi_2)} + \dots \quad (2)$$

where we will label the partial ref coeffs as  $\Gamma_i$  where

$$\Gamma_i \equiv \Gamma_{i,i+1} = \frac{\eta_{i+1} - \eta_i}{\eta_{i+1} + \eta_i} \quad (3)$$

To make this design process tractable, will assume:

- (i) all slabs are one-quarter wavelength long @ the design freq (though there will likely be of different physical length)
- (ii) the structure is symmetric s.t.  
 $\Gamma_0 = \Gamma_N, \Gamma_1 = \Gamma_{N-1}, \dots$

Hence, because of (i):

$$\Gamma_{in} \approx \Gamma_0 + \Gamma_1 e^{-j2\phi} + \Gamma_2 e^{-j4\phi} + \dots + \Gamma_N e^{-j2N\phi} \quad (5-74), (4)$$

$$\text{where } \phi = \beta_{n,i} t_n = \frac{2\pi}{\lambda_n} \cdot \left(\frac{\lambda_{n,0}}{4}\right) = \frac{\pi}{2} \left(\frac{f}{f_0}\right) \quad (5-74b), (5)$$

and  $f_0$  is the design frequency. At the design frequency  $f=f_0$ ,  $\phi = \frac{\pi}{2}$ .

Because of (ii), we can express (4) as

$$\Gamma_{in} \approx \Gamma_0 [1 + e^{-j2N\phi}] + \Gamma_1 [e^{-j2\phi} + e^{-j2(N-1)\phi}] + \dots \quad (6)$$

This sum has two forms depending if the number of layers is an even number or odd. We can write (6) as

$$\Gamma_{in} \approx e^{-jN\phi} \left\{ \Gamma_0 [e^{+jN\phi} + e^{-jN\phi}] + \Gamma_1 [e^{+j(N-2)\phi} + e^{-j(N-2)\phi}] + \text{Term} \right\} \quad (7)$$

$$\text{where Term} = \begin{cases} \Gamma_{(N-1)/2} (e^{+j\phi} + e^{-j\phi}) & N \text{ odd (even \# layers)} \\ \Gamma_{N/2} & N \text{ even} \end{cases}$$

Applying the cosine identity

$$\Gamma_{in}(f) \approx 2e^{-jN\phi} \left\{ \Gamma_0 \cos[N\phi] + \Gamma_1 \cos[(N-2)\phi] + \dots + \Gamma_n \cos[(N-2n)\phi] + \dots + \text{Term} \right\} \quad (5-15), (8)$$

where

$$\text{Term} = \begin{cases} \Gamma_{(N-1)/2} \cos \phi & N \text{ odd (even \# layers)} \\ \frac{1}{2} \Gamma_{N/2} & N \text{ even} \end{cases} \quad (9)$$

For symmetrical  
transformer.

One way to look at (8) is a Fourier cosine series expansion for the frequency variation of  $\Gamma_{in}(f)$ . This is true provided  $\Gamma_{in}$  is a real function (lossless media). The fundamental period in  $\phi$  of the periodic variation of  $\Gamma_{in}$  in (8), using (5), is:

$$\cos[(N-2n)\phi] \Rightarrow T_\phi = \frac{2\pi}{\omega_\phi} = \frac{2\pi}{2} = \pi$$

This corresponds to the frequency range over which the thickness of each slab changes by  $\lambda/2$ .

So, by properly choosing the  $\Gamma_n$  in (8) we can create a desired frequency variation in  $\Gamma_{in}$ , which was presumed to be small. We will consider two popular choices next.

However, it is instructive to compare this design of multi-layer slab structure with the design of multi-section quarter-wave matching transformers in microwave engineering. They're identical! (See Pozar, 3rd, sec. 5.5 - 5.7.)

## Binomial Design

in the binomial, (or maximally flat) design, the variation of  $\Gamma_{in}(f)$  wrt  $f$  near the design frequency is as small as possible. This occurs when the first  $N-1$  derivatives of  $|\Gamma_{in}|$  vanish, where  $N$  is the # of slab layers.

This type of response can be realized if we choose

$$\Gamma_{in}(f) = A (1 + e^{-j2\phi})^N \quad (10)$$

where

$$A = 2^{-N} \frac{\eta_L - \eta_0}{\eta_L + \eta_0} \quad (11)$$

It can be shown that this maximally flat design requires that the  $\Gamma_n$  in (8) be:

$$\underline{\Gamma_n} = A C_n^N \quad (5-77), (12)$$

where  $C_n^N$  are the binomial coefficients given by

$$C_n^N = \frac{N!}{(N-n)!n!} \quad n=0, 1, 2, \dots, N \quad (5-76a), (13)$$

Because this is a symmetrical structure,

$$C_n^N = C_{N-n}^N$$

Special values of the binomial coefficients include (recall that  $0! = 1! = 1$ ):

Because the impedances are related to the binomial coeffs through (1), this is called a binomial transformer design. This theory is approximate. The range of  $\eta_L$  is restricted to approximately  $\frac{\eta_0}{2} \leq \eta_L \leq 2\eta_0$  (14)

$$\frac{\eta_0}{2} \leq \eta_L \leq 2\eta_0 \quad (14)$$

- $C_0^N = \frac{N!}{(N-0)!0!} = \frac{N!}{N!} = 1$
- $C_1^N = \frac{N!}{(N-1)!1!} = \frac{N!}{(N-1)!} \cdot \frac{1}{N} = N \cdot \frac{N!}{N!} = N$
- $C_{N-1}^N = C_1^N = N$

With these  $\Gamma_n$  coefficients specified in (12), we're now in a position to determine the intrinsic impedances of every slab layer.   
 necessary

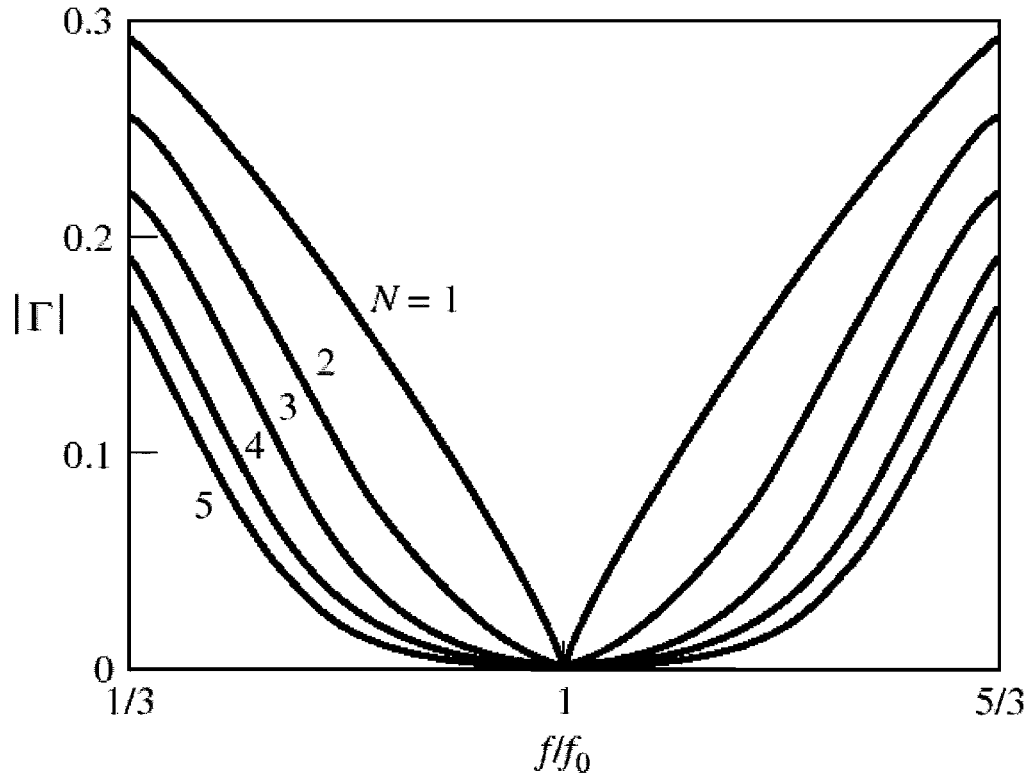
This can be accomplished using (3), presuming  $\eta_0$  &/or  $\eta_L$  are specified. This completes the analysis of the binomial design.   
 approximate transformer.

As given in the text, the fractional bandwidth for this binomial design is given as

$$\frac{\Delta f}{f_0} \approx 2 - \frac{4}{\pi} \cos^{-1} \left[ \frac{\Gamma_m}{(\eta_L - \eta_0)/(\eta_L + \eta_0)} \right]^{1/N} \quad (5-80), (15)$$

where  $\Gamma_m$  is the maximum ref. coeff. magnitude desired anywhere within the passband. <sup>One application of</sup> Eqn (15) can be used to determine the number of sections (N) needed to achieve a desired maximum ref. coeff mag ( $\Gamma_m$ ) over the bandwidth  $\Delta f$  centered @  $f_0$ .

[Pozar Fig. 5.15]



**Figure 5.15 (p. 250)**  
Reflection coefficient magnitude versus frequency for multisection binomial matching transformers of Example 5.6  $Z_L = 50\Omega$  and  $Z_0 = 100\Omega$ .

### Tschebyscheff (Chebyshev) Design

This design provides more bandwidth <sup>than the binomial transformer</sup> (with an equal number of layers) by increasing the amount of "ripple" of  $\Gamma_{in}$  in the passband.

This design is accomplished by equating the coeffs  $\Gamma_n$  in (8) w/ a Tschebyscheff polynomial;  $T_n(\cos \phi)$ .

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_n(x) = 2xT_{n-1} - T_{n-2}$$

← recurrence relation (numerically stable?)

These polynomials oscillate between  $\pm 1$  for  $x$  in the range  $|x| < 1$ . Outside this range they increase in magnitude w/o bound.

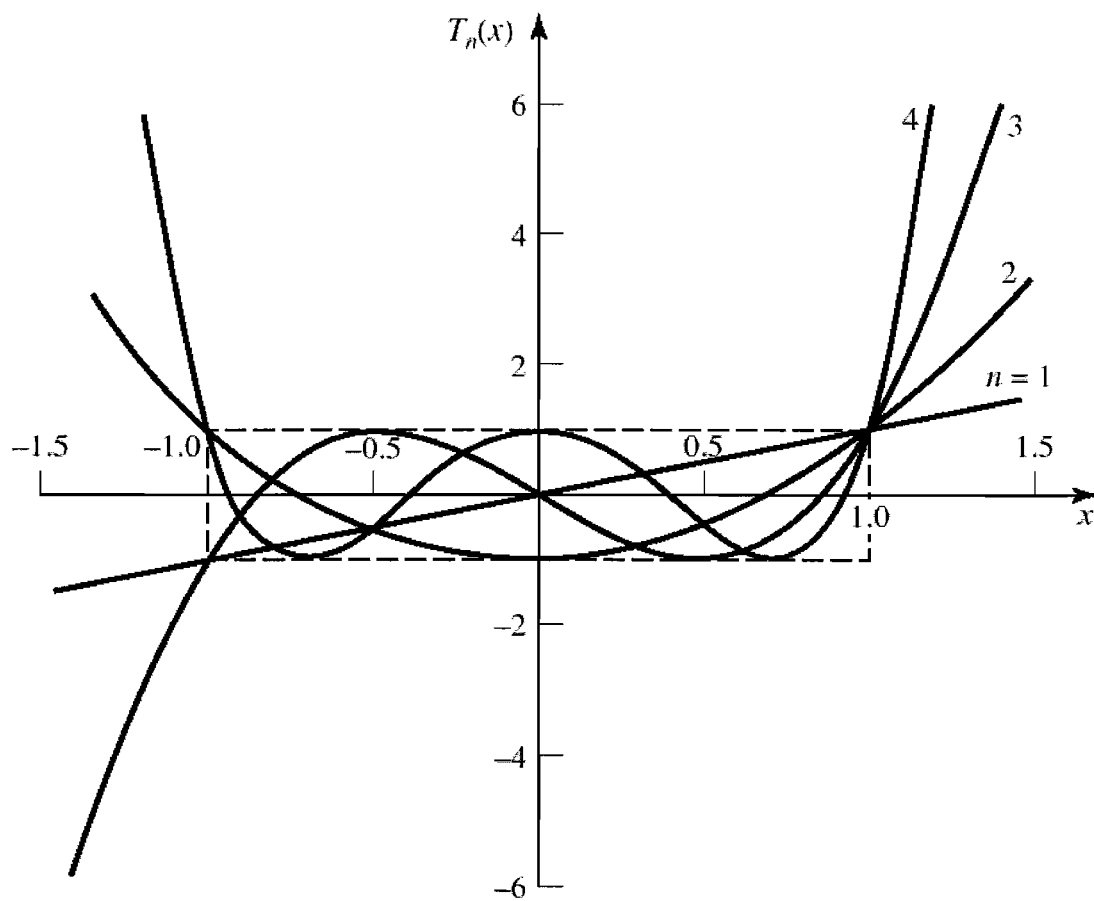
[Pozar Fig 5.16]

coeffs  $\Gamma_n$  in the

It can be shown that the approx  $\Gamma_{in}(f)$  in (8) should (somehow) be chosen s.t.

$$\Gamma_{in}(f) \approx e^{-jN\phi} \frac{\eta_L - \eta_0}{\eta_L + \eta_0} \frac{T_N(\sec \phi_m \cos \phi)}{T_N(\sec \phi_m)} \quad (5-81), (16)$$





**Figure 5.16 (p. 251)**

The first four Chebyshev polynomials  $T_n(x)$ .

where  $\phi_m$  ( $\dot{=} \pi - \phi_m$ ) are the edges of the passband when  $T_n(x) = \pm 1$ . At these frequencies,

$$\Gamma_m = \left| \frac{\eta_L - \eta_0}{\eta_L + \eta_0} \frac{1}{T_N(\sec \phi_m)} \right| \quad (5-82), (1)$$

There are no simple formulas for  $\Gamma_m$  in this design, though they can be determined by hand. (Assigned in your homework.)