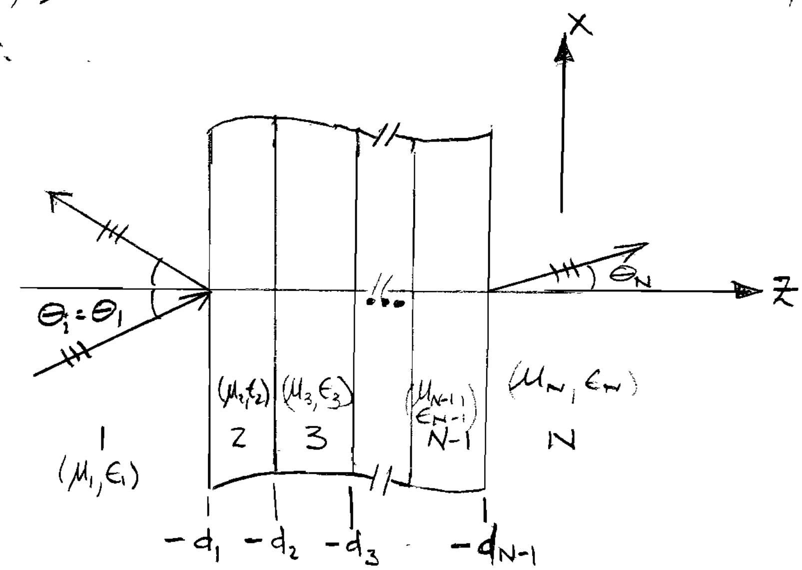
Lecture 15 – UPW Scattering by Multiple Slab Layers

Adding multiple slab layers is one technique for increasing the bandwidth of a low reflection surface. Multiple slabs are also one method of studying inhomogeneous media by finely dividing the media into layers of uniform material parameters.

In this lecture, we'll address the UPW scattering of this geometry:

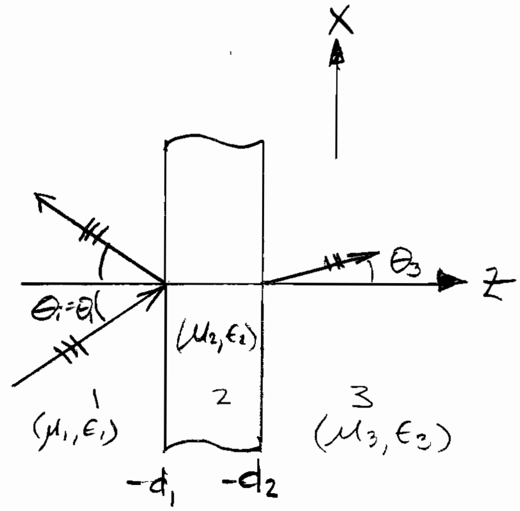


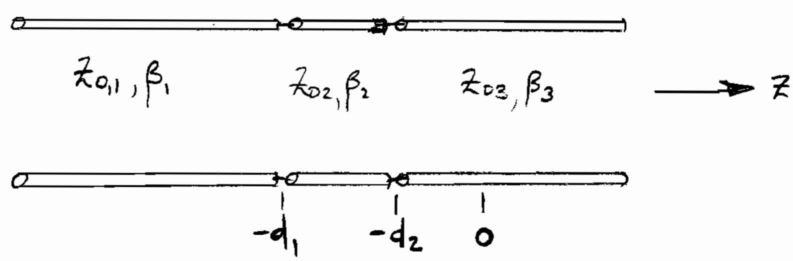
We will solve for the reflection & transmission coeffs. of ⊥ or ∥ pol UPWs incident in region 1.

There are a number of ways we can solve this problem. Perhaps the most straight-forward approach would be to cascade ABCD matrices in a TL analogous model for this problem.

We'll approach this solution using "constraint equations" which yield recursive equns for ref & trans. equations.

To begin,
Let's first re-consider the single layer (3 region) geometry:





The voltage wave in region 1 is

$$V_1(z) = A_1\left[e^{-j\beta_1 z} + \tilde{\Gamma}_{12}\, e^{j\beta_1 z} e^{j2\beta_1 d_1}\right] \qquad (1)$$

Where $\tilde{\Gamma}_{12}$ is a generalized reflection coefficient that is the ratio of the left-going wave amplitude to the right-going wave amplitude at the interface ($z = -d_1$). The extra phase factor $e^{j2\beta_1 d_1}$ ensures that this interpretation of $\tilde{\Gamma}_{12}$ is correct.

The wave in region 2 has a similar form

$$V_2(z) = A_2\left[e^{-j\beta_2 z} + \Gamma_{23}\, e^{j\beta_2 z} e^{j2\beta_2 d_2}\right] \qquad (2)$$

cbr (2), Γ_{23} is the ratio of the left-going wave amp. to the right going in region 2 at the interface $z=-d_2$. Because region 3 extends to infinity, Γ_{23} is the partial reflection coeff (Fresnel coeff)

$$\Gamma_{23} = \frac{Z_{0,3} - Z_{0,2}}{Z_{0,3} + Z_{0,2}} \qquad (3)$$

This is so because the entire left-going wave @ $z=-d_2$ is due only to the interface @ $z=-d_2$.

This fact also leads to this form for the wave in region 3

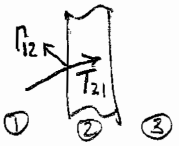
$$V_3(z) = A_3 e^{-j\beta_3 z} \qquad (4)$$

The unknowns A_1, A_2, A_3 & $\tilde{\Gamma}_{12}$ will be determined by imposing the so-called "constraint conditions" at the interfaces. These are simply self-consistent interrelationships between the right & left-going waves in each region.

For example, the right-going wave in region 2 is the consequence of the transmission of the right-going wave in region 1 as well as the reflection of the left-going wave in region 2.

Mathematically, at $z=-d_1$:

- From (2): $A_2 e^{j\beta_2 d_1}$ = right-going wave, region 2
- From (1): $T_{21} A_1 e^{j\beta_1 d_1}$ = transmission of right-going wave in region 1



(*) defined differently than Chew!

- From (2): $\Gamma_{21}\left(A_2\Gamma_{23}e^{-j\beta_2 d_1}e^{j2\beta_2 d_2}\right)$ reflection of left-going wave in region 2.

Combining these gives

$$\underline{A_2 e^{j\beta_2 d_1} = T_{21}A_1 e^{j\beta_1 d_1} + \Gamma_{21}A_2\Gamma_{23}e^{j(2\beta_2 d_2-\beta_2 d_1)}} \qquad (5)$$

Next, we observe that the left-going wave in region 1 is caused by the reflection of the right-going wave in region 1 plus the transmission of the left going wave in region 2. Therefore @ $z=-d_1$:

- From (1): $A_1\tilde{\Gamma}_{12}e^{-j\beta_1 d_1}e^{j2\beta_1 d_1}$ — left-going wave in region 1
- From (1): $\Gamma_{12}A_1 e^{+j\beta_1 d_1}$ — reflection of right-going wave in region 1.
- From (2): $T_{12}\left(A_2\Gamma_{23}e^{-j\beta_2 d_1}e^{j2\beta_2 d_2}\right)$ — transmission of left going wave in region 2.

Combining these gives

$$\underline{A_1\tilde{\Gamma}_{12}e^{j\beta_1 d_1} = \Gamma_{12}A_1 e^{j\beta_1 d_1} + T_{12}A_2\Gamma_{23}e^{j(2\beta_2 d_2-\beta_2 d_1)}} \qquad (6)$$

Now, to begin solution, From (5), $A_2\left[e^{j\beta_2 d_1} - \Gamma_{21}\Gamma_{23}e^{j(2\beta_2 d_2-\beta_2 d_1)}\right] = T_{21}A_1 e^{j\beta_1 d_1}$

or $$A_2 e^{j\beta_2 d_1}\left[1-\Gamma_{21}\Gamma_{23}e^{j2\beta_2(d_2-d_1)}\right] = T_{21}A_1 e^{j\beta_1 d_1}$$

s.t.

$$A_2 = \frac{T_{21}\,e^{j(\beta_1-\beta_2)d_1}}{1-\Gamma_{21}\Gamma_{23}e^{j2\beta_2(d_2-d_1)}}\,A_1 \qquad (7)$$

Next, substituting (7) into (6) we find that

$$A_1 \tilde{\Gamma}_{12} e^{j\beta_1 d_1} = \Gamma_{12} A_1 e^{j\beta_1 d_1} + \frac{T_{12} T_{21} \Gamma_{23} e^{j(\beta_1-\beta_2)d_1} e^{j(2\beta_2 d_2 - \beta_2 d_1)}}{1 - \Gamma_{21}\Gamma_{23} e^{j2\beta_2(d_2-d_1)}} A_1$$

For $A_1 \neq 0 \Rightarrow$

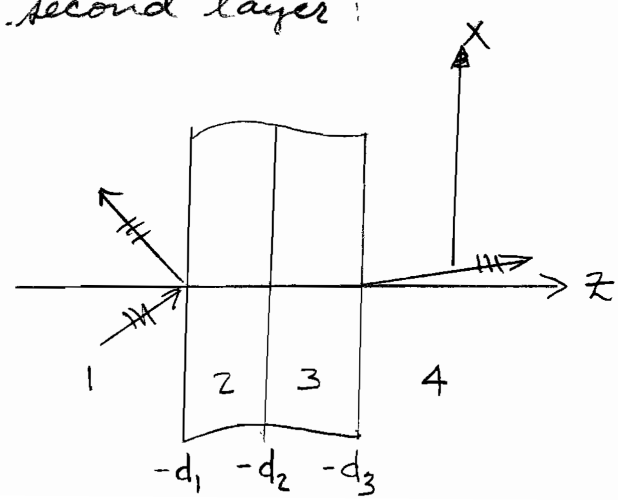
$$\tilde{\Gamma}_{12} = \Gamma_{12} + \frac{T_{12} T_{21} \Gamma_{23} e^{j2\beta_2(d_2-d_1)}}{1 - \Gamma_{21}\Gamma_{23} e^{j2\beta_2(d_2-d_1)}} \qquad \text{(one layer)} \qquad (8)$$

[Look familiar? Yes. Same as (10) in lecture 14!]

Chew, eqn (2.1.21), p. 51. though T_{ij} defined differently.

Referring back to (1), $\tilde{\Gamma}_{12}$ is a generalized ref. coef. that relates the total left-going wave @ $z=-d_1$ to the total right going wave. It includes all of the multiple reflections to the right of the $z=-d_1$ interface.

The power in this approach is that the effects of additional layers can be integrated recursively. For example, if we add a second layer:



If we carefully examine the previous derivation of $\tilde{\Gamma}_{12}$ in (8) we find that the in this two layer geometry we can use (8) with $\Gamma_{23} \longrightarrow \tilde{\Gamma}_{23}$:

$$\tilde{\Gamma}_{12} = \Gamma_{12} + \frac{T_{12} T_{21} \tilde{\Gamma}_{23} e^{j2\beta_2(d_2-d_1)}}{1-\Gamma_{21}\tilde{\Gamma}_{23} e^{j2\beta_2(d_2-d_1)}} \qquad (2 \text{ layers}) \qquad (9)$$

The reason is that in (8), Γ_{23} is the ratio of left- & right going waves in region 2. So, if a second layer (or more) is added, Γ_{23} becomes $\tilde{\Gamma}_{23}$.

How do we compute $\tilde{\Gamma}_{23}$? We can use a similar process that led to (8), except the media layers are different. If we replace $3 \rightarrow 4$, $2 \rightarrow 3$ & $1 \rightarrow 2$ in (8) we obtain

$$\tilde{\Gamma}_{23} = \Gamma_{23} + \frac{T_{23} T_{32} \Gamma_{34} e^{j2\beta_3(d_3-d_2)}}{1-\Gamma_{32}\Gamma_{34} e^{j2\beta_3(d_3-d_2)}} \qquad (10)$$

Since region 4 is infinite in extent, then Γ_{34} will be the partial ref. coeff (Fresnel coeff) for the 3-4 interface.

We can come up w/ a recursive formula for the generalized ref. coeff at the front face of a multi-layered slab structure:

$$\tilde{\Gamma}_{i,i+1} = \Gamma_{i,i+1} + \frac{T_{i,i+1} T_{i+1,i} \tilde{\Gamma}_{i+1,i+2} e^{j2\beta_{i+1}(d_{i+1}-d_i)}}{1-\Gamma_{i+1,i}\tilde{\Gamma}_{i+1,i+2} e^{j2\beta_{i+1}(d_{i+1}-d_i)}} \qquad (11)$$

For a recursive formula, need to express $\tilde{\Gamma}_{i,i+1}$ in terms of other $\tilde{\Gamma}$'s, rather than Γ's & T's.

Noting that $T_{ij} = 1 + \Gamma_{ji}$ & $\Gamma_{ij} = -\Gamma_{ji}$ then (11) becomes

$$\tilde{\Gamma}_{i,i+1} = \Gamma_{i,i+1} + \frac{(1-\Gamma_{i,i+1})(1-\Gamma_{i+1,i})\,\tilde{\Gamma}_{i+1,i+2}\, e^{j2\beta_{i+1}(d_{i+1}-d_i)}}{1 + \Gamma_{i,i+1}\tilde{\Gamma}_{i+1,i+2}\, e^{j2\beta_{i+1}(d_{i+1}-d_i)}} \qquad (12)$$

From the numerator

$$(1-\Gamma_{i,i+1})(1-\Gamma_{i+1,i}) = (1-\Gamma_{i,i+1})(1+\Gamma_{i,i+1}) = 1 - \Gamma_{i,i+1}^2$$

Sub. into (12):

$$\tilde{\Gamma}_{i,i+1} = \frac{\Gamma_{i,i+1} + \cancel{\Gamma_{i,i+1}^2\tilde{\Gamma}_{i+1,i+2}\, e^{j2\beta_{i+1}(d_{i+1}-d_i)}} + (1-\cancel{\Gamma_{i,i+1}^2})\tilde{\Gamma}_{i+1,i+2}\, e^{j2\beta_{i+1}(d_{i+1}-d_i)}}{\text{DEN}}$$

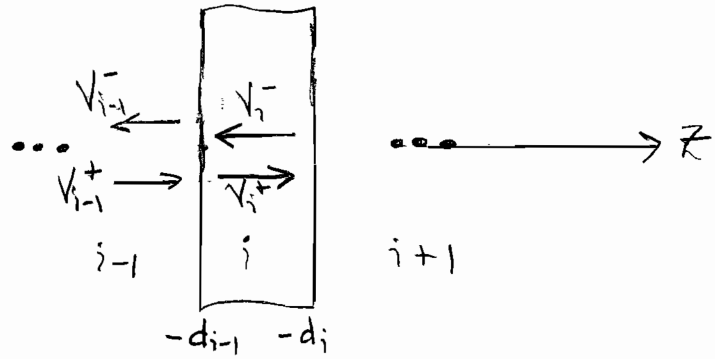
$$\boxed{\tilde{\Gamma}_{i,i+1} = \frac{\Gamma_{i,i+1} + \tilde{\Gamma}_{i+1,i+2}\, e^{j2\beta_{i+1}(d_{i+1}-d_i)}}{1 + \Gamma_{i,i+1}\tilde{\Gamma}_{i+1,i+2}\, e^{j2\beta_{i+1}(d_{i+1}-d_i)}}} \qquad (13)$$

← Chew, eqn (2.1.24) (even though T_{ij} defined differently)

Here we have $\tilde{\Gamma}_{i,i+1}$ expressed in terms of $\tilde{\Gamma}_{i+1,i+2}$. Beginning at the "right-most" interface, can recursive compute the ref. coeff at the front face. ($\tilde{\Gamma}_{N,N+1} = 0$)

Transmission Coefficient

We can derive an expression for the overall transmission coeff for the multislab geometry w/ oblique UPW illumination.



The right-going wave in region i is the sum of the reflection of the left-going wave in region i plus the transmission of the right-going wave in region $i-1$ [Somewhat similar to process leading to eqn (7).]

$$V_{i-1}(z) = A_{i-1}\left[e^{-j\beta_{i-1}z} + \tilde{\Gamma}_{i-1,i}\, e^{j\beta_{i-1}z}\, e^{j2\beta_i d_{i-1}}\right] \qquad (14)$$

$$V_i(z) = A_i\left[e^{-j\beta_i z} + \tilde{\Gamma}_{i,i+1}\, e^{j\beta_i z}\, e^{j2\beta_i d_i}\right] \qquad (15)$$

At $z = -d_{i-1}$:

- From (15): $A_i e^{j\beta_i d_{i-1}}$ = right going wave, region i.
- From (15): $\Gamma_{i,i-1} A_i \tilde{\Gamma}_{i,i+1} e^{-j\beta_i d_{i-1}} e^{j2\beta_i d_i}$ = reflection of left going wave in region i
- From (14): $T_{i,i-1} A_{i-1} e^{j\beta_{i-1} d_{i-1}}$ = transmission of right going wave in region $i-1$.

Using these, we can enforce the constraint condition at $z = -d_{i-1}$:

$$A_i e^{j\beta_i d_{i-1}} = \Gamma_{i,i-1} A_i \tilde{\Gamma}_{i,i+1} e^{-j\beta_i d_{i-1}} e^{j2\beta_i d_i} + T_{i,i-1} A_{i-1} e^{j\beta_{i-1} d_{i-1}} \qquad (16)$$

$$A_i e^{j\beta_i d_{i-1}} - \Gamma_{i,i-1} A_i \tilde{\Gamma}_{i,i+1} e^{-j\beta_i d_{i-1}} e^{j2\beta_i d_i} = T_{i,i-1} A_{i-1} e^{j\beta_{i-1} d_{i-1}}$$

$$A_i e^{j\beta_i d_{i-1}} \left[1 - \Gamma_{i,i-1} \tilde{\Gamma}_{i,i+1} e^{j2\beta_i (d_i - d_{i-1})} \right] = T_{i,i-1} A_{i-1} e^{j\beta_{i-1} d_{i-1}}$$

Therefore,

$$A_i e^{j\beta_i d_{i-1}} = \frac{T_{i,i-1} A_{i-1} e^{j\beta_{i-1} d_{i-1}}}{1 - \Gamma_{i,i-1} \tilde{\Gamma}_{i,i+1} e^{j2\beta_i (d_i - d_{i-1})}} \quad (17)$$

We'll define

Same S parameter as in microwave engineering.

$$\underline{\underline{S_{i,i-1} \equiv \frac{T_{i,i-1}}{1 - \Gamma_{i,i-1} \tilde{\Gamma}_{i,i+1} e^{j2\beta_i (d_i - d_{i-1})}}}} \quad (18)$$

So that (17) reads

$$\underline{\underline{A_i e^{j\beta_i d_{i-1}} = S_{i,i-1} A_{i-1} e^{j\beta_{i-1} d_{i-1}}}} \quad (19)$$

From (18) we can observe that all $S_{i,i-1}$ are now known since the $\tilde{\Gamma}_{i,i+1}$ can be computed recursively using (13). insert "A" So using (19) we can compute all A_i beginning w/ A_1, which is presumably known. More explictly, using (19) repeatedly:

$$i=2 : \quad A_2 e^{j\beta_2 d_1} = S_{21} A_1 e^{j\beta_1 d_1}$$

$$i=3 : \quad A_3 e^{j\beta_3 d_2} = S_{32} A_2 e^{j\beta_2 d_2} = S_{32} \left(A_2 e^{j\beta_2 d_1} \right) e^{j\beta_2 (d_2 - d_1)}$$

$$i=4 : \quad A_4 e^{j\beta_4 d_3} = S_{43} A_3 e^{j\beta_3 d_3} = S_{43} \left(A_3 e^{j\beta_3 d_2} \right) e^{j\beta_3 (d_3 - d_2)}$$

$$\vdots$$

$$i=N : \quad A_N e^{j\beta_N d_{N-1}} = S_{N,N-1} A_{N-1} e^{j\beta_{N-1} d_{N-1}} = S_{N,N-1} \left(A_{N-1} e^{j\beta_{N-1} d_{N-2}} \right) \cdot e^{j\beta_{N-1} (d_{N-1} - d_{N-2})}$$

Therefore, by repeatedly multiplying these results:

$$A_N e^{j\beta_N d_{N-1}} = S_{N,N-1} e^{j\beta_{N-1}(d_{N-1}-d_{N-2})} \cdots S_{43} e^{j\beta_3(d_3-d_2)} \cdot$$

$$S_{32} e^{j\beta_2(d_2-d_1)} \cdot S_{21} A_1 e^{j\beta_1 d_1}$$

or

$$\underline{A_N e^{j\beta_N d_{N-1}} = A_1 e^{j\beta_1 d_1} \cdot \prod_{n=1}^{N-1} S_{n+1,n} e^{j\beta_n(d_n-d_{n-1})}} \qquad (20)$$

where we define $\underline{\underline{d_0 \equiv d_1}}$ & $d_N \equiv d_{N-1}$ → to "B"

"A"

To determine an expression for the overall transmission factor, it is helpful to write down the fields in regions 1 & N. From (15):

$$V_1(z) = A_1 \left[e^{-j\beta_1 z} + \tilde{\Gamma}_{12}\, e^{j\beta_1 z} e^{j2\beta_1 d_1} \right] \qquad (21)$$

&

$$V_N(z) = A_N e^{-j\beta_N z} \qquad (22)$$

The overall transmission factor will be the ratio

$$\tilde{T}_{N1} \equiv \frac{V_N^+(z=-d_{N-1})}{V_1^+(z=-d_1)}$$

Substituting from (21) & (22):

$$\tilde{T}_{N1} = \frac{A_N e^{+j\beta_N d_{N-1}}}{A_1 e^{j\beta_1 d_1}} \qquad (23)$$

"B" Substituting from (20)

$$\tilde{T}_{N1} = \prod_{n=1}^{N-1} S_{n+1,n}\, e^{j\beta_n (d_n - d_{n-1})} \qquad (24)$$

Similar to Chew, eqn (2.1.28)

$(d_0 \equiv d_1)$

Here focusing on UPWs. Same math for microwave engineering → Multi-section impedance matching network.