To this point, we've assumed both half spaces were lossless. To one extent or another, all materials are lossy so it will be useful for us to consider the effects of loss. We'll accomplish this by studying UPW illumination of a lossy half space. In particular, region 1 will be lossless and region 2 will be lossy:

![Diagram of UPW illumination with lossy region]  

There are multiple sources for energy loss of an EM wave. Here we're considering just electric conduction. This energy loss causes the EM amplitude to decrease as it propagates through the lossy material, converting it to heat.

As you saw in Ch. 3 and in your homework, the wave equation for $E$ is

$$\nabla^2 E - \kappa^2 E = 0$$  \hspace{1cm} (3.37) \hspace{1cm} (1)

where  \[ \kappa^2 = j \omega \mu (\tau + j \omega \varepsilon) \]  \hspace{1cm} (3.37d) \hspace{1cm} (2)

or  \[ \kappa = \pm \sqrt{j \omega \mu (\tau + j \omega \varepsilon)} = \pm (\alpha + j \beta) \]  \hspace{1cm} (3.37e) \hspace{1cm} (3)

solutions for $\alpha$ and $\beta$ given in (4.28).
...This leads to solutions of the form $e^{+\alpha z \pm i\beta z}$.

Specifically, for the geometry shown above:

\[ E^i = \hat{x} E_0 e^{-i\beta z}, \quad E^r = \hat{x} E_0 \Gamma^b e^{+i\beta z}, \quad E^t = \hat{x} E_0 T^b e^{-\alpha z \pm i\beta z} \quad (4) \]

and

\[ H^i = \frac{\hat{\eta}_1}{\hat{\eta}_2} E_0 e^{-i\beta z}, \quad H^r = -\frac{\hat{\eta}_1}{\hat{\eta}_2} E_0 \Gamma^b e^{+i\beta z}, \quad H^t = \frac{\hat{\eta}_1}{\hat{\eta}_2} E_0 T^b e^{-\alpha z \pm i\beta z} \quad (5) \]

The coefficients $\Gamma^b$ and $T^b$ will now be complex as well.

$\eta_2$ and $\alpha$ are real (positive) numbers.

However,

Enforcing b.c.'s at the interface, we find $\Gamma^b$ and $T^b$ are the same as for lossless materials:

\[ \Gamma^b = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{and} \quad T^b = \frac{2\eta_2}{\eta_2 + \eta_1} \quad (5-50), \]

\[ \eta_2 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad (6) \]

So, there is not much difference to their solutions than the lossless case.

---

Skein Depth

One consequence of this is the EM field decays exponentially as we see in $E^t \approx H^t$ in (4) & (5).

Since $\mathcal{F}_c = \sigma \mathcal{E}$, then for the half space geometry in the previous figure:

\[ \mathcal{F}_{c/2} (z) = \sigma_2 \mathcal{E}^t \approx \frac{\hat{x}}{\sigma_2} E_0 T^b e^{-\alpha z \pm i\beta z} \quad (7) \]
Considering the magnitude of the current density

\[ |J_{x,c}(z)| = |E_0T^b e^{-\frac{z^2}{\delta^2} e^{-\sqrt{\kappa}z}}| = |E_0T^b e^{-\frac{z^2}{\delta^2}}| \]  

(8)

If we call \( J_0 = |J_{x,c}(z=0)| = |E_0T^b| \)  

(9)

then we can express (8) as

\[ |J_{x,c}(z)| = J_0 e^{-\frac{z^2}{\delta^2}} \]

(10)

This current density appears as

The distance into the conductor at which the magnitude of \( J_0 \) has been reduced by the factor \( e^{-1} \) is called the skin depth, \( \delta \), mentioned briefly in Lecture 6.

\[ \delta_2 = \frac{1}{\alpha_2} = \left\{ \frac{\omega \mu \epsilon_0}{2 \left( \sqrt{1 + \left( \frac{\alpha_2}{\omega \epsilon_0} \right)^2} - 1 \right)} \right\}^{-\frac{1}{2}} \]  

\[
(\text{m})
\]

(11) 

If region 2 is a "good" conductor (\( \sigma_2 \gg \omega \epsilon_2 \))

\[ \delta_2 \approx \frac{1}{\sqrt{\sigma_2 \omega}} \]  

\[
[\text{m}]
\]

(4.43) 

(12)
This skin depth can be very small for metals at high frequencies. For example, consider Au with $\nu = 4.1 \times 10^7 \ (\mu = \mu_0)$:

<table>
<thead>
<tr>
<th>$f$</th>
<th>$\delta = \sqrt{\frac{\mu_0 \mu_0 \nu}{\pi}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 MHz</td>
<td>24.9 $\mu$m</td>
</tr>
<tr>
<td>100 MHz</td>
<td>7.86 $\mu$m</td>
</tr>
<tr>
<td>1 GHz</td>
<td>2.49 $\mu$m</td>
</tr>
<tr>
<td>10 GHz</td>
<td>0.79 $\mu$m</td>
</tr>
<tr>
<td>30 GHz</td>
<td>0.45 $\mu$m</td>
</tr>
</tbody>
</table>

For $z < 5 \delta_z$, the EM field can be considered essentially zero.

A physical interpretation for skin depth is illustrated in the text on p. 209:

The integral over all current density magnitudes is:

$$|J_S| = \int_0^\infty |J_{x,c}| \, dz = \int_0^\infty |J_0| e^{-z/\delta_z} \, dz = -\delta_z |J_0| e^{-\frac{z}{\delta_z}}$$

$$|J_S| = \delta_z |J_0|$$

In other words, the magnitude of the current density per unit width is equal to that at the surface considered constant to a distance $\delta_z$. Integrating