Whites

Lecture 12 – Critical Angle of Incidence

For certain ∢'s of incidence, it is possible for $\Gamma_\perp^b$ to become 1, even though not at grazing incidence. We observe this behavior when $\epsilon_1 > \epsilon_2$ ($\mu_1 = \mu_2$) in the results of lecture 9.

Odd behavior since: ✓ $|\Gamma_\perp^b| = 1$ and $\measuredangle \Gamma_\perp^b \neq 0, 180°$
✓ $|T_\perp^b|$ ranges from 2 to 0 as $\theta_i \to 90°$ (grazing)

So, the wave is totally reflected, but $|T_\perp^b| \neq 0$ which seems to imply that fields are not zero in region 2. Strange!

We'll investigate this phenomenon in much detail. In lecture 9, showed θ_t vs. θ_i for $\epsilon_{r1} = 4$, $\epsilon_{r2} = 1$ ($\mu_{r1} = \mu_{r2} = 1$). At approx. 30°, θ_t becomes 90°. For larger θ_i, θ_t becomes imaginary!! What does that mean? See soon.

Compute θ_i when $\theta_t = 90°$. From Snell's law of refraction

$$\theta_t = \sin^{-1}\left(\frac{\beta_1}{\beta_2}\sin\theta_i\right) \qquad (1)$$

For $\theta_t = \frac{\pi}{2} \Rightarrow \sin\left(\frac{\pi}{2}\right) = \frac{\sqrt{\mu_1\epsilon_1}}{\sqrt{\mu_2\epsilon_2}}\sin\theta_c$ ← called critical angle of incidence

or

$$\theta_c = \sin^{-1}\left(\sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}\right) \qquad (5\text{-}35b),(2)$$

The argument of the $\sin^{-1}$ fct. cannot exceed 1 (for real θ_c) then this requires for the existence of a critical ∢ that

$$\mu_2\epsilon_2 \le \mu_1\epsilon_1 \qquad (5\text{-}35c),(3)$$

Now, if $\mu_1 = \mu_2$ then in order for a critical ∢ to exist $\epsilon_2 < \epsilon_1$. In other words, a critical ∢ of incidence requires that EM wave traveling from more "dense" EM medium to a less dense one.

Can easily show that

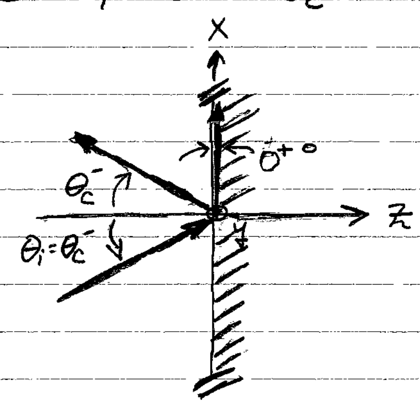
$$\Gamma_\perp^b\Big|_{\theta_i=\theta_c} = 1 \quad ; \quad T_\perp^b\Big|_{\theta_i=\theta_c} = 2 \qquad (3),(4)$$

For the example results @ the end of this lecture, $\frac{\epsilon_1}{\epsilon_2} = 4$; $\frac{\mu_1}{\mu_2} = 1$.
From (2),

$$\underline{\theta_c} = \sin^{-1}\left(\sqrt{\frac{1}{4}}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \underline{\underline{30^\circ}}$$

From the plot, $|\Gamma_\perp^b| = 1$; $\measuredangle \Gamma_\perp^b = 0$ @ $\theta_i = 30^\circ$ as predicted. Also, $|T_\perp^b| = 2$; $\measuredangle T_\perp^b = 0$.

This is a very interesting behavior. We also see that $|\Gamma_\perp^b| = 1$ for $\theta_i \geq \theta_c$. What is happening here? To understand this, take a look at the plot of θ_t. It increases rapidly from 0 @ $\theta_i = 0$ to 90° @ $\theta_i = 30^\circ = \theta_c$. At the critical $\measuredangle$ of incidence:



From (1), $\theta_t\Big|_{\theta_i=\theta_c} = \sin^{-1}\left[\frac{\sqrt{\mu_1\epsilon_1}}{\sqrt{\mu_2\epsilon_2}}\sin\theta_i\right]_{\theta_i=\theta_c} = \sin^{-1}\left[\sqrt{4}\,\sin 30^\circ\right] = \sin^{-1}[1]$

So, $\theta_t = 90^\circ$ as we know. Now, what happens to θ_t when $\theta_i > \theta_c$? The argument of sin is > 1, so there is no longer a real angle of transmission. Instead, θ_t becomes a complex angle.

What does that mean? No geometrical interpretation. Rather, it is much more meaningful to work directly w/ the wavevector $\bar{\beta}^t$. As we saw earlier

$$\beta_x^t = \beta_x^i \quad ; \qquad (4)$$

$$\beta_z^t = \beta_2 \cos\theta_t = \pm\sqrt{\beta_2^2 - \beta_x^{i\,2}} \qquad (5)$$

At the critical angle of incidence, $\theta_t = 90^\circ \underset{(5)}{\Rightarrow} \beta_z^t = 0$ (6)

We deduce from (5) that @ $\theta_i = \theta_t$, $(\beta_x^i)^2 = \beta_2^2$. (7)
For $\theta_i < \theta_c$, then θ_t is real so that $(\beta_x^i)^2 < \beta_2^2$ while for $\theta_i > \theta_c$ then $(\beta_x^i)^2 > \beta_2^2$ so that θ_t is complex.
In other words:

because field not uniform in planes ⊥ direction of prop (+x) in region 2. Still UPW in region 1.

$$\left.\begin{array}{lll} \theta_i < \theta_c : & (\beta_x^i)^2 < \beta_2^2 & \text{s.t. } \beta_z^t \text{ real} \\ \theta_i = \theta_c : & (\beta_x^i)^2 = \beta_2^2 & \text{s.t. } \beta_z^t = 0 \\ \theta_i > \theta_c : & (\beta_x^i)^2 > \beta_2^2 & \text{s.t. } \beta_z^t \text{ imag.} \end{array}\right\} \qquad (8)$$

In this later case, $\bar{E}_\perp^t = \hat{y}\, T_\perp^b E_o e^{-j\bar{\beta}^t \cdot \bar{r}}$ (9)

or
$$\bar{E}_\perp^t = \hat{y}\, T_\perp^b E_o e^{-j(\beta_x^t x + \beta_z^t z)} = \hat{y}\, T_\perp^b E_o e^{-j\beta_x^i x} e^{-j\beta_z^t z} \qquad (10)$$

Therefore, with $\theta_i > \theta_c$, then from (5) $\beta_z^t = -j\alpha_z^t = -j\sqrt{(\beta_x^i)^2 - \beta_2^2}$ (11)

so that (10) becomes

The PW propagates in x but now is purely attenuated in z because $\theta_i > \theta_c$. Fig 5-6. Non-uniform plane wave

$$\bar{E}_\perp^t = \hat{y}\, T_\perp^b E_o e^{-j\beta_x^i x} e^{-\alpha_z^t z} \quad \frac{V}{m} \qquad (12)$$

Non

The wave continues to propagate along the interface w/ phase velocity
$$v_{px} = \frac{\omega}{\beta_x^i} \qquad (13)$$

From (8) & (13), we can deduce that

$$\left.\begin{array}{ll} \theta_i < \theta_c : & v_{px} > v_2 \\ \theta_i = \theta_c : & v_{px} = v_2 = \dfrac{1}{\sqrt{\mu_2 \epsilon_2}} \\ \theta_i > \theta_c : & v_{px} < v_2 \end{array}\right\} \qquad (14)$$

In this latter case, called a slow wave

Surface waves appear in many EM applications, such as dielectric wgds., patch antennas & microwave ckts, certain antennas (& freqs.) located near the Earth, etc.

A wave that propagates along an interface but decays away from it, as (12), is often called a surface wave. Depending on β_x^i, this attenuation can happen very quickly so that the field is tightly bound to the interface.
in z

Combining all of these terms, we could have a tightly bound slow surface wave, depending on β_x^i.

Referring back to the beginning of this section, we were puzzled by results that showed for certain ∡'s of incidence of a UPW incident on a less-dense half space $|\Gamma_\perp^b| = 1$ & $|T_\perp^b| \neq 0$. Seemed to violate our concept of conservation of power.

Let's check this. Your text shows that @ $\theta_i = \theta_c$:

$$\bar{S}_{AV}^t = \frac{1}{2} Re\left\{\bar{E}_\perp^t \times \bar{H}_\perp^{t*}\right\}\Big|_{\theta_i=\theta_c} = \hat{x}\,\frac{2|E_0|^2}{\eta_2} \quad [W/m^2] \qquad (5\text{-}40), (15)$$

Notice there is time-avg. power flow in x direction, but none in the z direction (normal to interface)! Interesting.

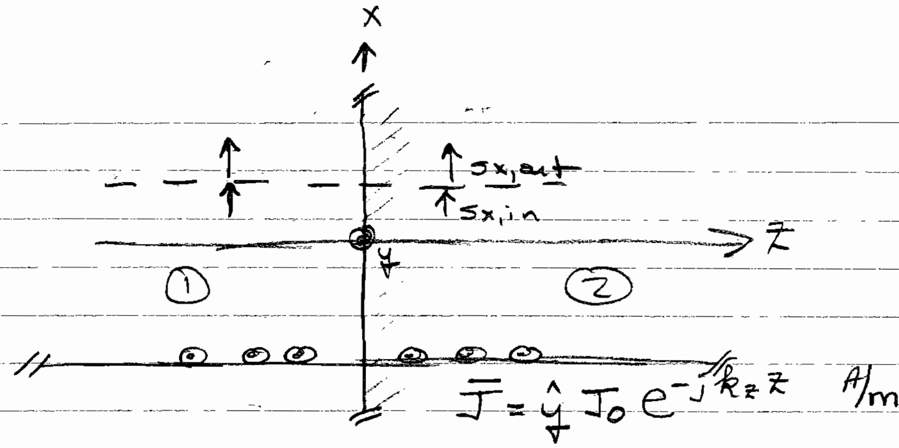
Your text also shows that

$$\bar{S}_{AV}^i = \frac{1}{2} Re\left\{\bar{E}_\perp^i \times \bar{H}_\perp^{i*}\right\}\Big|_{\theta_i=\theta_c} = (\hat{x}\sin\theta_c + \hat{z}\cos\theta_c)\,\frac{|E_0|^2}{2\eta_1} \quad (5\text{-}41a), (16)$$

$$\bar{S}_{AV}^r = \frac{1}{2} Re\left\{\bar{E}_\perp^r \times \bar{H}_\perp^{r*}\right\}\Big|_{\theta_i=\theta_c} = (\hat{x}\sin\theta_c - \hat{z}\sin\theta_c)\,\frac{|E_0|^2}{2\eta_1} \quad (5\text{-}41b), (17)$$

We see here that $S_{z,AV}^i = -S_{z,AV}^r$ meaning all time-avg. power incident in z direction is reflected back. Consistent w/ (5-40).

Do we expect that $S_{x,AV}^t \stackrel{?}{=} S_{x,AV}^i \;(= S_{x,AV}^r)$? To answer this question, perhaps we can consider this geometry: (18)



Answer is: not necessarily. We do know that for any plane ⊥ to x, $S_{x,in} = S_{x,out}$.

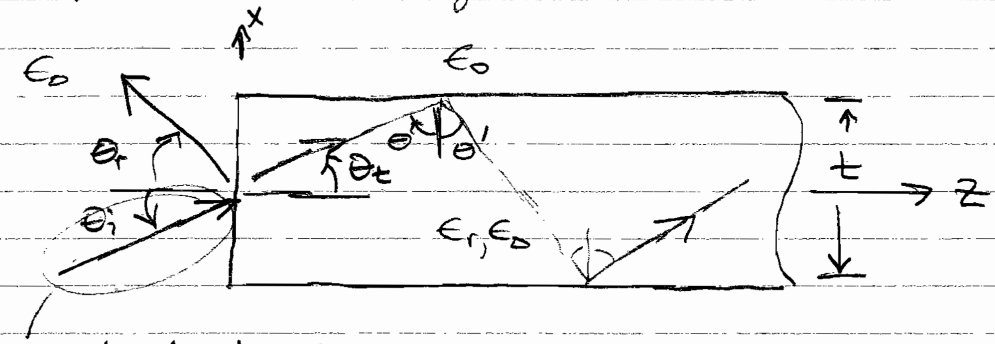
Even though fields in region 2 are not zero for $\theta_i \geq \theta_c$, there is no time avg. power flow in z (normal to interface). There is, however, time-avg. power flow in x (parallel to interface).

Critical Angle : Parallel-Pol

Besides particular field expressions, the results we've derived here concerning the critical ∡ of incidence, power flow = 0 normal to interface in region 2 for $\theta_i \geq \theta_c$, non-uniform plane waves, surface waves, slow waves, etc. are exactly the same for a ∥-pol UPW obliquely incident on a half space.

This makes sense because Snell's laws are identical for these UPW polarizations. We derived θ_c & other phenomenon from these laws.

Example N12.1 (Text example 5-6). Classic problem.
A light ray is incident on a dielectric slab as shown.
Determine range of ϵ_r so that light is contained in slab and propagates down this waveguide.



A "ray" is a tightly bundled UPW w/ a field that is contained in a "tube".
Tube diameter << t

In order to contain ray, $\theta' \geq \theta_c \underset{\uparrow (2)}{=} \sin^{-1}\left(\sqrt{\frac{\epsilon_0}{\epsilon_r\epsilon_0}}\right)$ (19)

which means $\theta_t \leq \frac{\pi}{2} - \theta' = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{\sqrt{\epsilon_r}}\right)$

or $\sin\left(\frac{\pi}{2} - \theta_t\right) \geq \frac{1}{\sqrt{\epsilon_r}}$ or $\cos\theta_t \geq \frac{1}{\sqrt{\epsilon_r}}$ (20)

By phase match @ the interface $z=0$, $\left(-\frac{t}{2} \leq x \leq \frac{t}{2}\right)$

$$\cancel{\beta_0}\sin\theta_i = \cancel{\beta_0}\sqrt{\epsilon_r}\sin\theta_t$$

$$\Rightarrow \sin\theta_t = \frac{1}{\sqrt{\epsilon_r}}\sin\theta_i \qquad (21)$$

By definition $\cos^2\theta_t = 1 - \sin\theta_t^2$

or

$$\cos^2\theta_t = \pm\sqrt{1 - \sin\theta_t^2} \qquad (22)$$

Square (20) & sub (22):

$$\cos^2\theta_t \geq \frac{1}{\epsilon_r} \quad \text{or} \quad 1 - \sin\theta_t^2 \geq \frac{1}{\epsilon_r}$$

and sub (21):

$$\epsilon_r - \epsilon_r\left(\frac{1}{\epsilon_r}\sin^2\theta_i\right) \geq 1$$

$$\therefore \quad \epsilon_r \geq 1 + \sin^2\theta_i$$

To accomodate all θ_i, then minimum ϵ_r is

$$\epsilon_r \geq 1 + \max\left[\sin^2\theta_i\right]_{\theta_i} = 2 \quad \text{when } \theta_i = 90°.$$

When $\theta_i = 0$, all wave will prop. in z & be contained in dielectric wgd. As θ_i increases, ϵ_r increases. More reflection @ front face, however.

[DEMO w/ laser pointer
& glass of water.]