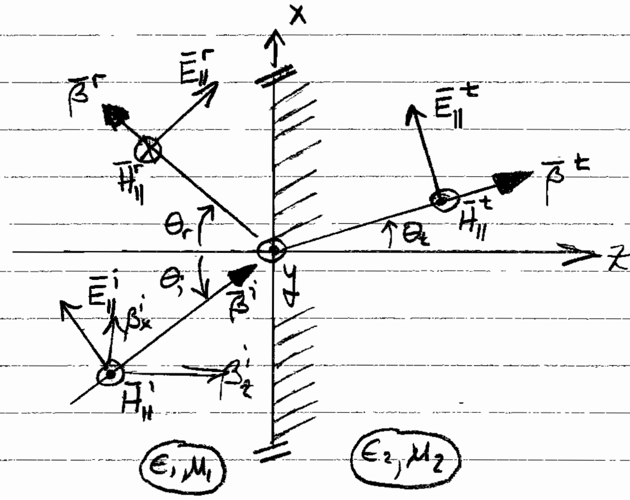
Lecture 10: UPWs Obliquely Incident on a Lossless Half Space: H Pol.

This is the other principal polarization discussed in the previous lecture. For this case, we'll assume the electric & magnetic fields are polarized as (Fig. 5-4):



The choice of these directions for $\bar{E}^i$, $\bar{E}^r$ & $\bar{E}^t$ were made so that at $\theta_i = 0$, all $\bar{E}$'s will be pointing in the same direction. So we'll expect the Γ & τ for ∥-pol. to equal those of our previous work @ normal incidence. @ $\theta_i = 0$

The $\bar{E}$ & $\bar{H}$ fields are then

$$\bar{H}_{\parallel}^i = \hat{y}\frac{E_0}{\eta_1} e^{-j\bar{\beta}^i\cdot\bar{r}} \text{ A/m} \qquad \text{(5-20b), (1)}$$

& from $\nabla\times\bar{H} = +j\omega\epsilon\bar{E} \underset{e^{-j\bar{\beta}^i\cdot\bar{r}}}{\Rightarrow} -j\bar{\beta}\times\bar{H}^i = j\frac{\beta_1}{\eta_1}\bar{E}^i$

or
$$\underline{\bar{E}^i} = -\frac{\eta_1}{\beta_1}\bar{\beta}^i\times\bar{H}^i = \underline{-\eta_1\hat{\beta}^i\times\bar{H}^i} \qquad (2)$$

then
$$\bar{E}^i = -\eta_1\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \sin\theta_i & 0 & \cos\theta_i\\ 0 & H_y^i & 0\end{vmatrix} = -\eta_1\left[\hat{x}(-\cos\theta_i H_y^i) + \hat{z}\sin\theta_i H_y^i\right]$$

or
$$\bar{E}^i = (\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)E_0 e^{-j\bar{\beta}^i\cdot\bar{r}} \qquad \text{(5-20a), (3)}$$

where $\bar{\beta}^i\cdot\bar{r} = \beta_x^i x + \beta_z^i z = \beta_1(x\sin\theta_i + z\cos\theta_i)$

Similarly, $\bar{H}_{\parallel}^{r} = -\hat{y}\frac{\Gamma_{\parallel}^{b}}{\eta_1} E_o e^{-j\bar{\beta}^{r}\cdot\bar{r}}$ A/m (5-21b), (4)

and

$$\bar{E}_{\parallel}^{r} = -\eta_1 \hat{\beta}^{r} \times \bar{H}^{r} = (\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)\,\Gamma_{\parallel}^{b} E_o e^{-j\bar{\beta}^{r}\cdot\bar{r}} \ \frac{V}{m},$$
$$= (\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)\, E_{\parallel}^{r} e^{-j\bar{\beta}^{r}\cdot\bar{r}}$$

w/ $\bar{\beta}^{r}\cdot\bar{r} = \beta_x^{r} x - \beta_z^{r} z = \beta_1(x\sin\theta_r - z\cos\theta_r)$ (5-21a), (5)

While $\bar{H}_{\parallel}^{t} = \hat{y}\frac{T_{\parallel}^{b}}{\eta_2} E_o e^{-j\bar{\beta}^{t}\cdot\bar{r}}$ A/m (5-22b), (6)

and

$$\bar{E}_{\parallel}^{t} = -\eta_2 \bar{\beta}^{t} \times \bar{H}^{t} = (\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)\, T_{\parallel}^{b} E_o e^{-j\bar{\beta}^{t}\cdot\bar{r}} \ \frac{V}{m},$$
$$= (\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)\, E_{\parallel}^{t} e^{-j\bar{\beta}^{t}\cdot\bar{r}}$$

w/ $\bar{\beta}^{t}\cdot\bar{r} = \beta_x^{t} x + \beta_z^{t} z = \beta_2(x\sin\theta_t + z\cos\theta_t)$ (5-22a), (7)

The reflection & transmission coeffs. are defined wrt the <u>total</u> Electric field amplitude as

$$\Gamma_{\parallel}^{b} \equiv \left.\frac{E_{\parallel}^{r}}{E_{\parallel}^{i}}\right|_{z=0} \quad \& \quad T_{\parallel}^{b} \equiv \left.\frac{E_{\parallel}^{t}}{E_{\parallel}^{i}}\right|_{z=0} \qquad (8)$$

rather than wrt a component of $\bar{E}$ normal or tangential to the interface. A slight subtlety.

To determine the coefficients $\Gamma_{\parallel}^{b}$ & $T_{\parallel}^{b}$, we apply the boundary conditions on the tangetial fields:

- <u>E_{tan} continuous</u>: For the $\hat{x}$ components of $\bar{E}$ @ $z=0$ using (3), (5) and (7):

$$\cos\theta_i E_o e^{-j\beta_x^{i} x} + \cos\theta_r \Gamma_{\parallel}^{b} E_o e^{-j\beta_x^{r} x} = \cos\theta_t T_{\parallel}^{b} E_o e^{-j\beta_x^{t} x} \qquad \text{(5-23a), (9)}$$

In order for (9) to be valid for all x coordinates, then

$$\underline{\beta_x^{i} = \beta_x^{r} = \beta_x^{t}} \qquad (10)$$

This is the phase match condition once again. From (10) we find that

✔ $\beta_x^i = \beta_x^r \Rightarrow \beta_1 \sin\theta_i = \beta_1 \sin\theta_r$ s.t. $\underline{\theta_i = \theta_r}$ (5-24a), (11)
This is Snell's law of reflection, once again.

✔ $\beta_x^i = \beta_x^t \Rightarrow \underline{\beta_1 \sin\theta_i = \beta_2 \sin\theta_t}$ (5-24b), (12)
which is Snell's law of refraction.

Comparing (11) & (12) with similar equations in the previous lecture we see that both principal polarizations ($\perp$ & $\parallel$) have the same Snell's laws.

So, using (11) in (9) we find

$$\cos\theta^i \left(1 + \Gamma_\parallel^b\right) = \cos\theta_t \, T_\parallel^b \tag{13}$$

- $\underline{\bar{H}_{tan} \text{ continuous}}$: For the $\hat{y}$ components of $\bar{H}$ @ $z=0$ using (1), (4) and (6)

$$\frac{E_0}{\eta_1} e^{-j\beta_x^i x} - \frac{\Gamma_\parallel^b}{\eta_1} E_0 e^{-j\beta_x^r x} = \frac{T_\parallel^b}{\eta_2} E_0 e^{-j\beta_x^t x} \qquad \text{(5-23b), (14)}$$

Employing the phase match condition from (10) reduces this equation to

$$\frac{1}{\eta_1}\left(1 - \Gamma_\parallel^b\right) = \frac{T_\parallel^b}{\eta_2} \tag{14}$$

Solving (13) & (14) simultaneously gives

$$\frac{\eta_2}{\eta_1} = \frac{\sqrt{\frac{\mu_2}{\epsilon_2}}}{\sqrt{\frac{\mu_1}{\epsilon_1}}} = \frac{\sqrt{\epsilon_1 \epsilon_2}}{\sqrt{\mu_1 \mu_2}}$$

$$\boxed{\Gamma_\parallel^b = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}} = \frac{\frac{\eta_2}{\eta_1}\cos\theta_t - \cos\theta_i}{\frac{\eta_2}{\eta_1}\cos\theta_t + \cos\theta_i} \qquad \text{(5-24c), (15)}$$

and

$$\boxed{T_\parallel^b = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}} \frac{\frac{1}{\eta_1}}{\frac{1}{\eta_1}} = \frac{2\frac{\eta_2}{\eta_1}\cos\theta_i}{\frac{\eta_2}{\eta_1}\cos\theta_t + \cos\theta_i} \qquad \text{(5-24d), (16)}$$

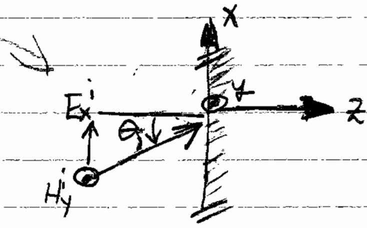
TL Analogy

A TL analogy can be developed for $\parallel$-pol UPWs incident on planar interfaces, as we did w/ $\perp$-pol UPWs. Examining $T_\parallel^b$ in (16), though, we see a problem. While η_1 & $\cos\theta_i$ appear together in (15) & (16), η_2 & $\cos\theta_i$ appear together in (16).

It turns out that the origin of this difficulty is with the definition of $\Gamma_\parallel^b$ & $T_\parallel^b$. We used the total $\bar{E}$ rather than a component transverse to the interface normal.

Referring to the following figure, for this $\parallel$-pol,

$$Z_{+z,1} \equiv \frac{E_x^i}{H_y^i} \tag{17}$$



and

$$Z_{+z,2} \equiv \frac{E_x^t}{H_y^t} \tag{18}$$

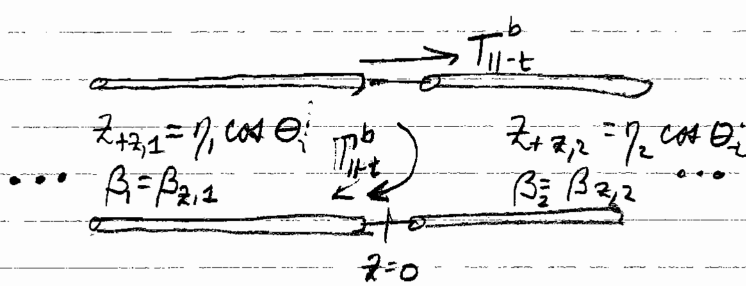
From (17) & substituting from (3) & (1) we find

$$Z_{+z,1} = \frac{\cos\theta_i E_0}{E_0/\eta_1} = \eta_1 \cos\theta_i \tag{19}$$

& from (18) w/ (4) & (5).

$$Z_{+z,2} = \frac{\cos\theta_t \, T_\parallel^b E_0}{T_\parallel^b E_0/\eta_2} = \eta_2 \cos\theta_t \tag{20}$$

Consequently, the TL analogous model for $\parallel$-pol scattering by a planar interface is



So the reflection & Transmission coeffs for the transverse electric field (E_x in this case) is

$$\Gamma_{\parallel\text{-}t}^{b} \equiv \left.\frac{E_x^r}{E_x^i}\right|_{z=0} = \frac{Z_{+z,2} - Z_{+z,1}}{Z_{+z,2} + Z_{+z,1}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{21}$$

and

$$T_{\parallel\text{-}t}^{b} \equiv \left.\frac{E_x^t}{E_x^i}\right|_{z=0} = \frac{2 Z_{+z,2}}{Z_{+z,2} + Z_{+z,1}} = \frac{2\eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{22}$$

So, we can use this TL analogous model to solve $\parallel$-pol problems. If we wish to know $\Gamma_{\parallel}^{b}$ & $T_{\parallel}^{b}$, these can easily be computed from (21) & (22).

From (5) we see that
$$\left.E_x^r\right|_{z=0} = \cos\theta_r \, \Gamma_{\parallel}^{b} E_0 e^{-j\beta_x^i x} \tag{23}$$
while from (3)
$$\left.E_x^i\right|_{z=0} = \cos\theta_i \, E_0 e^{-j\beta_x^r x} \tag{24}$$

w/ $\beta_x^r = \beta_x^i \Rightarrow \theta_r = \theta_i$ then dividing (23) & (24) gives

$$\frac{\left.E_x^r\right|_{z=0}}{\left.E_x^i\right|_{z=0}} = \Gamma_{\parallel}^{b} \Rightarrow \underline{\Gamma_{\parallel\text{-}t}^{b} = \Gamma_{\parallel}^{b}} \tag{25}$$

From (7) we see that
$$\left.E_x^t\right|_{z=0} = \cos\theta_t \, T_{\parallel}^{b} E_0 e^{-j\beta_x^t x} \tag{26}$$

w/ $\beta_x^t = \beta_x^i$ then dividing (26) by (24) gives

$$\frac{\left.E_x^t\right|_{z=0}}{\left.E_x^i\right|_{z=0}} = \frac{\cos\theta_t}{\cos\theta_i} T_{\parallel}^{b} \Rightarrow \underline{T_{\parallel\text{-}t}^{b} = \frac{\cos\theta_t}{\cos\theta_i} T_{\parallel}^{b}} \tag{27}$$

— Results — $\boxed{\text{Fig 5-5}}$

✓ Replot w/o mag.
✓ Plot $T_{\parallel\text{-}t}^{b}$
✓ Plot $Z_{+z,1}$ & $Z_{+z,2}$

Duality & Dual Solutions

Section 7.2 in your text describes duality in Maxwell's equations. We can develop this concept by comparing similar Maxwell's equations to each other: From Table 1-4:

$\nabla \times \bar{E} = -\bar{M}_i - J_{c,m} - j\omega\mu\bar{H}$	$\nabla \times \bar{H} = \bar{J}_i + \bar{J}_{c,e} + j\omega\epsilon\bar{E}$
$\nabla \cdot \bar{D} = \rho_{e,v}$	$\nabla \cdot \bar{B} = \rho_{m,v}$
$\nabla \cdot \bar{J}_{c,e} = -j\omega\rho_{e,v}$	$\nabla \cdot \bar{J}_{c,m} = -j\omega\rho_{m,v}$
$\bar{J}_{c,e} = \sigma_e \bar{E}$	$\bar{J}_{c,m} = \sigma_m \bar{H}$

These two sets of equations have identical forms, just the symbols are different. A simple way to construct one set of these equations from the other is:

- ✓ Replace all electric field quantities w/ dual magnetic ones
- ✓ Replace all magnetic field quantities w/ negative of dual electric ones
- ✓ Interchange dual electric & magnetic material
- ✓ parameters ($\mu \longleftrightarrow \epsilon$, which implies $\eta \longleftrightarrow \frac{1}{\eta}$, $\sigma_e \longleftrightarrow \sigma_m$).

Dual field Quantities

$\bar{E}$	$\bar{H}$
$\bar{D}$	$\bar{B}$
$\bar{J}_i$	$\bar{M}_i$
$\bar{J}_{c,e}$	$\bar{J}_{c,m}$
ρ_{ev}	ρ_{em}

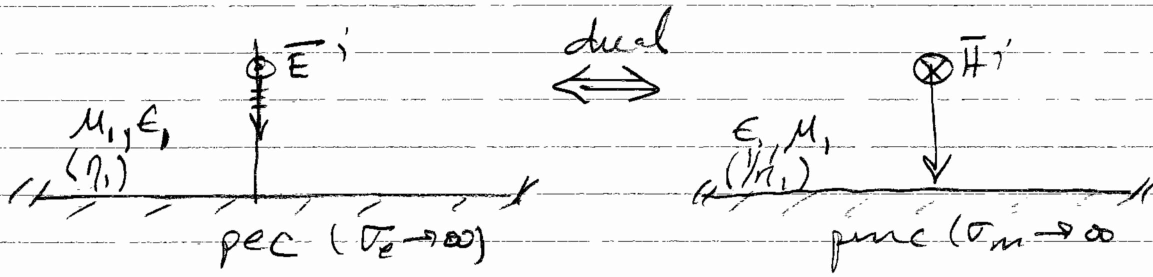
Dual Material Parameters

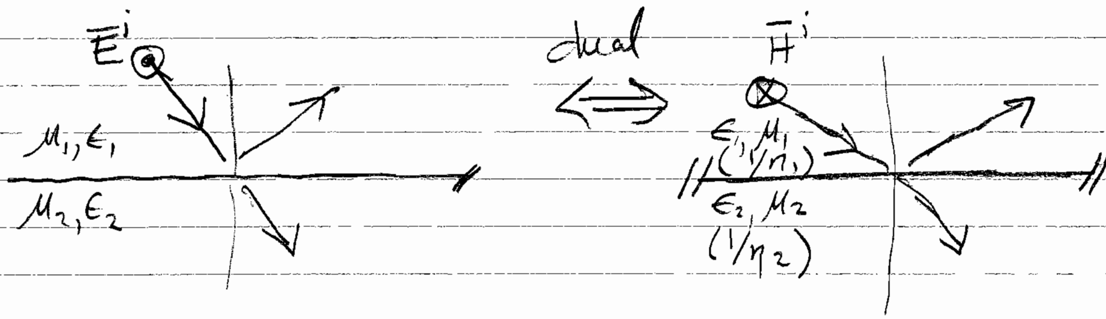
ϵ	μ
σ_e	σ_m

This can be a quick check of a problem, or a simple method of constructing equations of one problem from a dual if the latter field is known, but the former is not.

Additionally, if a problem is dual to another & the boundary conditions are also dual, then the full solutions for one problem will be dual to the other. (Your text doesn't distinguish between duality & dual solutions.) This can be extremely useful.

Examples of problems w/ dual solutions

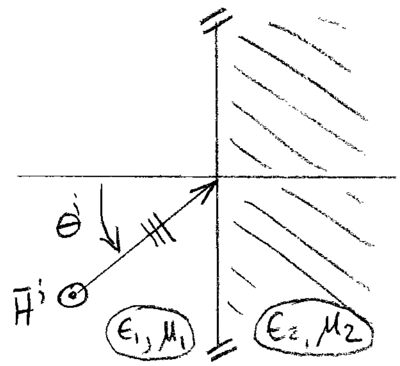




$\boxed{\text{Example N10.1}}$

Parallel Polarized UPW Obliquely Incident on a Lossless Half Space - 1

EE 692 Advanced Engineering Electromagnetics
Keith W. Whites



In[100]:=

```
<< Graphics`Legend`
```

In[101]:=

$\frac{\epsilon_1}{\epsilon_2} = \frac{1}{4}$

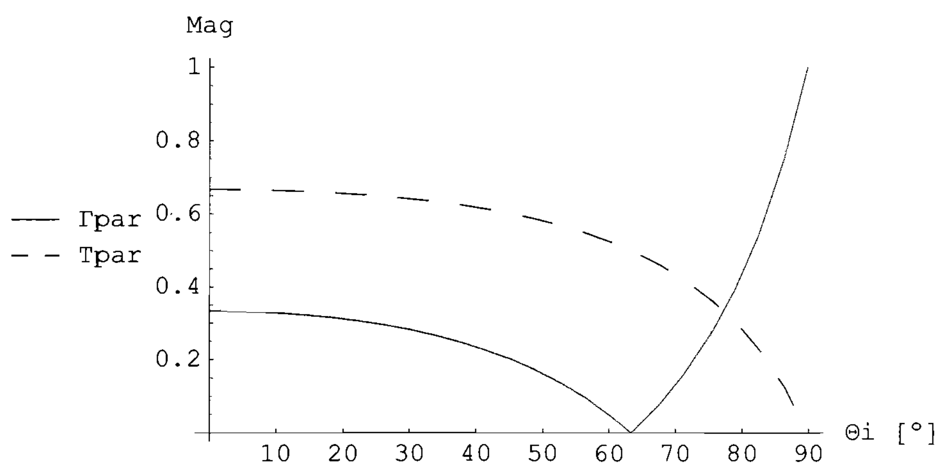
```
ε1overε2 := 1 / 4
μ1overμ2 := 1 / 1
```

$\mu_1 = \mu_2$

```
θt[θi_] := ArcSin[Sqrt[μ1overμ2 * ε1overε2] * Sin[θi]]
Γpar[θi_] := (Sqrt[ε1overε2 / μ1overμ2] * Cos[θt[θi]] - Cos[θi]) /
  (Sqrt[ε1overε2 / μ1overμ2] * Cos[θt[θi]] + Cos[θi])
Tpar[θi_] := 2 * Sqrt[ε1overε2 / μ1overμ2] *
  Cos[θi] / (Sqrt[ε1overε2 / μ1overμ2] * Cos[θt[θi]] + Cos[θi])
```

In[106]:=

```
Plot[{Abs[Γpar[θi * Degree]], Abs[Tpar[θi * Degree]]}, {θi, 0, 90},
 PlotRange → Automatic, Ticks → {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
 PlotStyle → {{RGBColor[1, 0, 0]}, {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},
 AxesLabel → {"θi  [°]", "Mag"}, PlotLegend → {"Γpar", "Tpar"},
 LegendSize → {0.5, 0.2}, LegendShadow → None, LegendPosition → {-1.4, -0.1}]
```



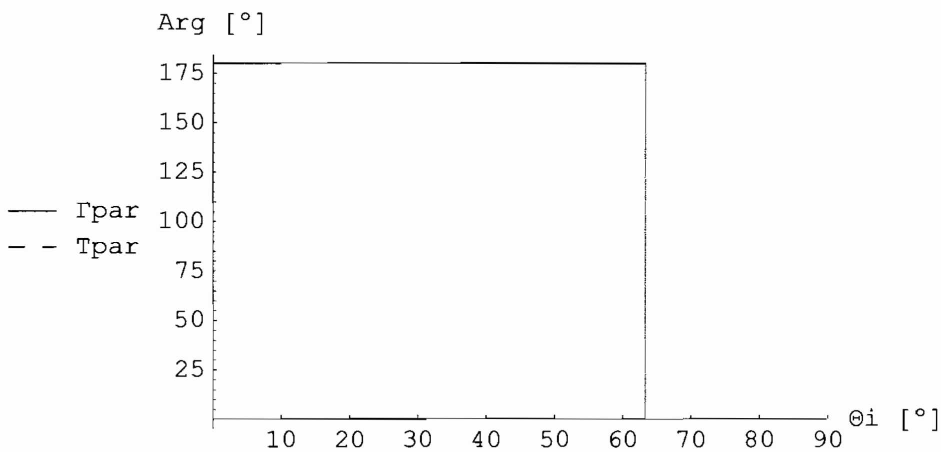
Out[106]=

```
- Graphics -
```

$\boxed{\text{Example N10.1 (cont.)}}$

In[107]:=

```
Plot[{Arg[Γpar[θi * Degree]] / Degree, Arg[Τpar[θi * Degree]] / Degree}, {θi, 0, 90},
 PlotRange → {{0, 90}, Automatic}, Ticks → {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
 PlotStyle → {{RGBColor[1, 0, 0]}, {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},
 AxesLabel → {"θi [°]", "Arg [°]"}, PlotLegend → {"Γpar", "Τpar"},
 LegendSize → {0.5, 0.2}, LegendShadow → None, LegendPosition → {-1.4, -0.1}]
```

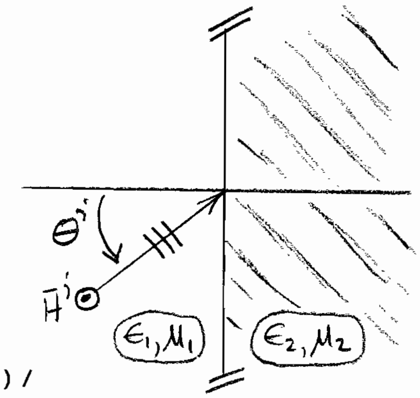


Out[107]=

- Graphics -

Example N10.2

Parallel Polarized UPW Obliquely Incident on a Lossless Half Space - 1



EE 692 Advanced Engineering Electromagnetics
Keith W. Whites

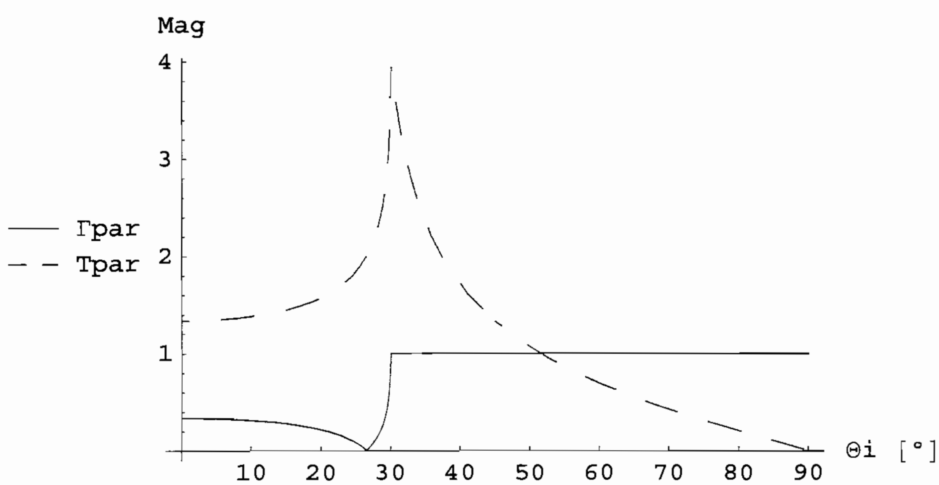
```
In[10]:= << Graphics`Legend`
```

$\frac{\epsilon_1}{\epsilon_2} = 4$

```
In[11]:= ε1overε2 := 4 / 1
         μ1overμ2 := 1 / 1
```

$\mu_1 = \mu_2$

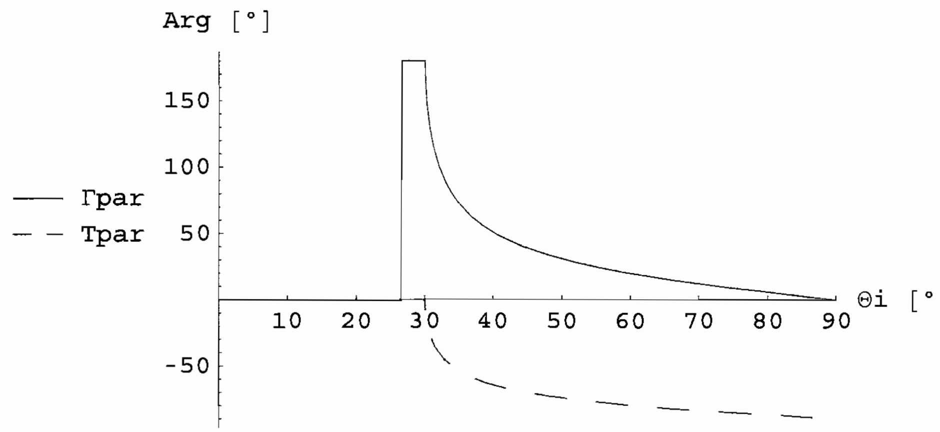
```
         θt[θi_] := ArcSin[Sqrt[μ1overμ2 * ε1overε2] * Sin[θi]]
         Γpar[θi_] := (Sqrt[ε1overε2 / μ1overμ2] * Cos[θt[θi]] - Cos[θi]) /
            (Sqrt[ε1overε2 / μ1overμ2] * Cos[θt[θi]] + Cos[θi])
         Tpar[θi_] := 2 * Sqrt[ε1overε2 / μ1overμ2] *
            Cos[θi] / (Sqrt[ε1overε2 / μ1overμ2] * Cos[θt[θi]] + Cos[θi])

In[16]:= Plot[{Abs[Γpar[θi * Degree]], Abs[Tpar[θi * Degree]]}, {θi, 0, 90},
          PlotRange → Automatic, Ticks → {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
          PlotStyle → {{RGBColor[1, 0, 0]}, {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},
          AxesLabel → {"θi  [°]", "Mag"}, PlotLegend → {"Γpar", "Tpar"},
          LegendSize → {0.5, 0.2}, LegendShadow → None, LegendPosition → {-1.4, -0.1}]
```




```
Out[16]= - Graphics -
```

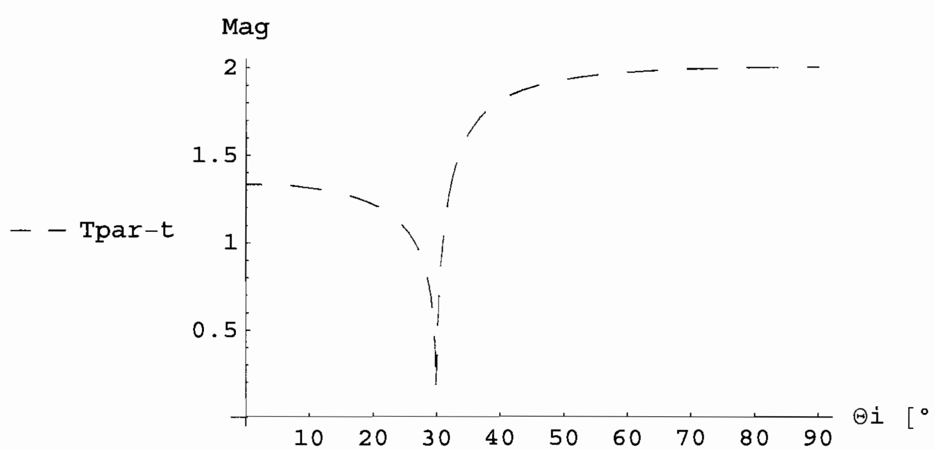
Example N10.2 (cont.)

```
In[17]:= Plot[{Arg[Γpar[θi*Degree]] / Degree, Arg[Tpar[θi*Degree]] / Degree},
   {θi, 0, 90}, PlotRange → {{0, 90}, Automatic},
   Ticks → {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
   PlotStyle → {{RGBColor[1, 0, 0]}, {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},
   AxesLabel → {"θi [°]", "Arg [°]"}, PlotLegend → {"Γpar", "Tpar"},
   LegendSize → {0.5, 0.2}, LegendShadow → None, LegendPosition → {-1.4, -0.1}]
```




```
Out[17]= - Graphics -
```

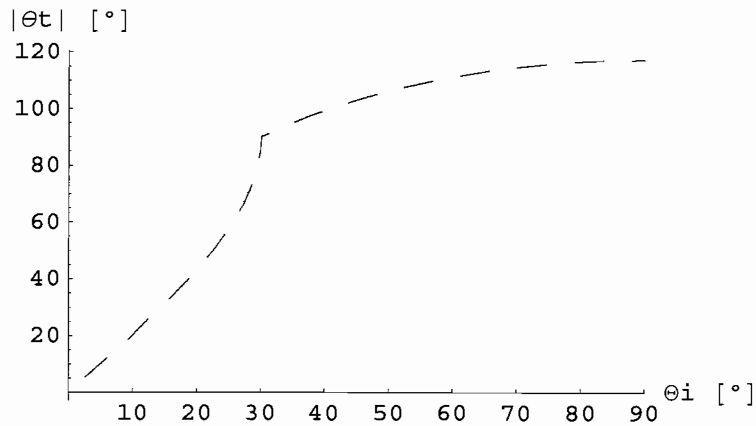
```
In[18]:= Plot[Abs[Tpar[θi*Degree]*Cos[θt[θi*Degree]] / Cos[θi*Degree]], {θi, 0, 90},
   PlotRange → Automatic, Ticks → {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
   PlotStyle → {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]},
   AxesLabel → {"θi [°]", "Mag"}, PlotLegend → {"Tpar-t"},
   LegendSize → {0.5, 0.2}, LegendShadow → None, LegendPosition → {-1.5, -0.1}]
```




```
Out[18]= - Graphics -
```

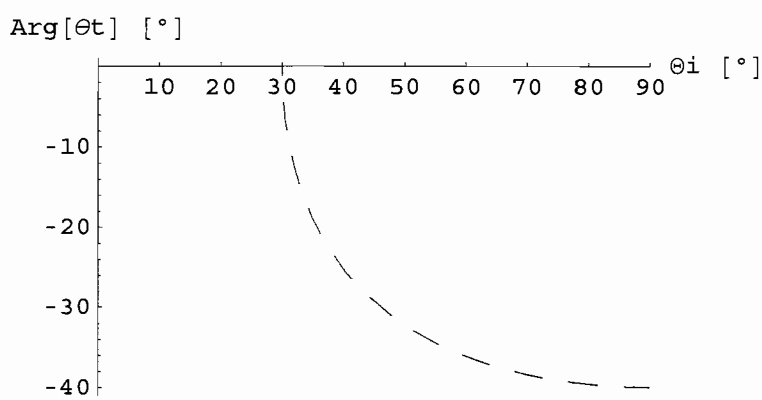
Example N10.2 (cont.)

In[19]:=
```
Plot[Abs[θt[θi*Degree] / Degree], {θi, 0, 90}, PlotRange → {{0, 90}, Automatic},
  Ticks → {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
  PlotStyle → {{Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},
  AxesLabel → {"θi [°]", "|θt| [°]"}]
```



Out[19]= - Graphics -

```
Plot[Arg[θt[θi*Degree] / Degree] / Degree, {θi, 0, 90}, PlotRange → {{0, 90}, Automatic},
  Ticks → {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
  PlotStyle → {{Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},
  AxesLabel → {"θi [°]", "Arg[θt] [°]"}]
```



Out[20]= - Graphics -