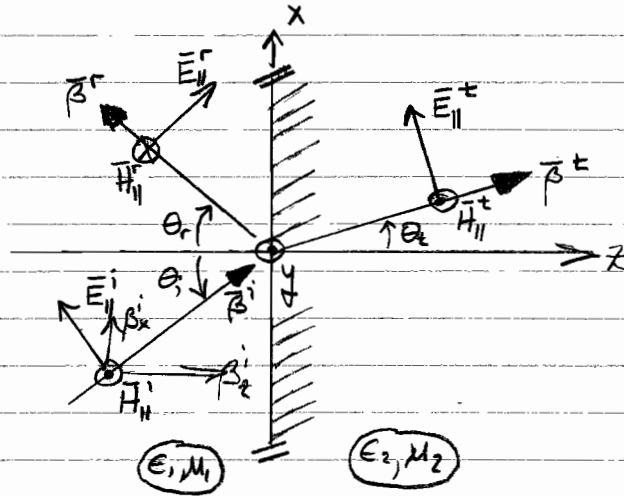


This is the other principal polarization discussed in the previous lecture. For this case, we'll assume the electric & magnetic fields are polarized as (Fig. 5-4):



The choice of these directions for  $\vec{E}^i$ ,  $\vec{E}^r$  &  $\vec{E}^t$  were made so that at  $\theta_i = 0$ , all  $\vec{E}$ 's will be pointing in the same direction. So we'll expect the  $\Gamma$  &  $\tau$  for H-pol to equal those of our previous work @ normal incidence. @  $\theta_i = 0$

The  $\vec{E}$  &  $\vec{H}$  fields are then

$$\vec{H}_{||}^i = \hat{y} \frac{E_0}{\eta_1} e^{-j\vec{\beta}^i \cdot \vec{r}} \quad \text{H-pol} \quad (5-20b), (1)$$

from  $\nabla \times \vec{H} = +j\omega \epsilon \vec{E} \Rightarrow \frac{\partial}{\partial x} \vec{H}^i = -j\vec{\beta}^i \times \vec{H}^i = j \frac{\mu}{\eta_1} \vec{E}^i$

$$\text{or } \vec{E}^i = -\frac{\eta_1}{\beta_1} \vec{\beta}^i \times \vec{H}^i = -\eta_1 \hat{\beta}^i \times \vec{H}^i \quad (2)$$

then

$$\vec{E}^i = -\eta_1 \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta_i & 0 & \cos\theta_i \\ 0 & H_y^i & 0 \end{vmatrix} = -\eta_1 \left[ \hat{x} (-\cos\theta_i H_y^i) + \hat{z} \sin\theta_i H_y^i \right]$$

$$\text{or } \vec{E}^i = (\hat{x} \cos\theta_i - \hat{z} \sin\theta_i) E_0 e^{-j\vec{\beta}^i \cdot \vec{r}} \quad (5-20a), (3)$$

where  $\vec{\beta}^i \cdot \vec{r} = \beta_x^i x + \beta_z^i z = \beta_1 (x \sin\theta_i + z \cos\theta_i)$

Similarly,  $\vec{H}_{\parallel}^r = -\hat{y} \frac{\eta_{\parallel}^b}{\eta_1} E_0 e^{-j\vec{\beta}^r \cdot \vec{r}} \quad A/m \quad (5-21b), (4)$

and

$$\vec{E}_{\parallel}^r = -\eta_1 \hat{\beta}^r \times \vec{H}^r = (\hat{x} \cos \theta_r + \hat{z} \sin \theta_r) \eta_{\parallel}^b E_0 e^{-j\vec{\beta}^r \cdot \vec{r}} \frac{V}{m}$$

$$= (\hat{x} \cos \theta_r + \hat{z} \sin \theta_r) E_{\parallel}^r e^{-j\vec{\beta}^r \cdot \vec{r}}$$

w/  $\vec{\beta}^r \cdot \vec{r} = \beta_x^r x - \beta_z^r z = \beta_1 (x \sin \theta_r - z \cos \theta_r) \quad (5-21a), (5)$

while  $\vec{H}_{\parallel}^t = \hat{y} \frac{T_{\parallel}^b}{\eta_2} E_0 e^{-j\vec{\beta}^t \cdot \vec{r}} \quad A/m \quad (5-22b), (6)$

and

$$\vec{E}_{\parallel}^t = -\eta_2 \vec{\beta}^t \times \vec{H}^t = (\hat{x} \cos \theta_t - \hat{z} \sin \theta_t) T_{\parallel}^b E_0 e^{-j\vec{\beta}^t \cdot \vec{r}} \frac{V}{m}$$

$$= (\hat{x} \cos \theta_t - \hat{z} \sin \theta_t) E_{\parallel}^t e^{-j\vec{\beta}^t \cdot \vec{r}}$$

w/  $\vec{\beta}^t \cdot \vec{r} = \beta_x^t x + \beta_z^t z = \beta_2 (x \sin \theta_t + z \cos \theta_t) \quad (5-22a), (7)$

The reflection & transmission coeffs. are defined w/ the total electric field amplitudes as

$$\Gamma_{\parallel}^b \equiv \left. \frac{E_{\parallel}^r}{E_{\parallel}^i} \right|_{z=0} \quad ; \quad T_{\parallel}^b \equiv \left. \frac{E_{\parallel}^t}{E_{\parallel}^i} \right|_{z=0} \quad (8)$$

rather than w/ a component of  $\vec{E}$  normal or tangential to the interface. A slight subtlety.

To determine the coefficients  $\Gamma_{\parallel}^b$  &  $T_{\parallel}^b$ , we apply the boundary conditions on the tangential fields:

•  $\vec{E}_{tan}$  continuous: For the  $\hat{x}$  components of  $\vec{E}$  @  $z=0$  using (3), (5) and (7): (5-23a),

$$\cos \theta_i E_0 e^{-j\beta_x^i x} + \cos \theta_r \Gamma_{\parallel}^b E_0 e^{-j\beta_x^r x} = \cos \theta_t T_{\parallel}^b E_0 e^{-j\beta_x^t x} \quad (9)$$

In order for (9) to be valid for all  $x$  coordinates, then

$$\underline{\beta_x^i = \beta_x^r = \beta_x^t} \quad (10)$$

This is the phase match condition once again. From (10) we find that

$$\checkmark \beta_x^i = \beta_x^r \Rightarrow \beta_1 \sin \theta_i = \beta_1 \sin \theta_r \quad \text{s.t.} \quad \underline{\theta_i = \theta_r} \quad (5-24a), (11)$$

This is Snell's law of reflection, once again.

$$\checkmark \beta_x^i = \beta_x^t \Rightarrow \beta_1 \sin \theta_i = \beta_2 \sin \theta_t \quad (5-24b), (12)$$

which is Snell's law of refraction.

Comparing (11) & (12) with similar equations in the previous lecture we see that both principal polarizations ( $\perp$  &  $\parallel$ ) have the same Snell's laws.

So, using (11) in (9) we find

$$\cos \theta_i (1 + \Gamma_{\parallel}^b) = \cos \theta_t T_{\parallel}^b \quad (13)$$

•  $\vec{H}_{\text{tan}}$  continuous: For the  $\hat{y}$  components of  $\vec{H} @ z=0$  using (1), (4) and (6)

$$\frac{E_0}{\eta_1} e^{-j\beta_1^i x} - \frac{\Gamma_{\parallel}^b}{\eta_1} E_0 e^{-j\beta_1^r x} = \frac{T_{\parallel}^b}{\eta_2} E_0 e^{-j\beta_2^t x} \quad (5-23b), (14)$$

Employing the phase match condition from (10) reduces this equation to

$$\frac{1}{\eta_1} (1 - \Gamma_{\parallel}^b) = \frac{T_{\parallel}^b}{\eta_2} \quad (14)$$

Solving (13) & (14) simultaneously gives

$$\frac{\eta_2}{\eta_1} = \frac{\sqrt{\mu_2/\epsilon_2}}{\sqrt{\mu_1/\epsilon_1}} = \frac{\sqrt{\epsilon_1/\epsilon_2}}{\sqrt{\mu_1/\mu_2}}$$

$$\Gamma_{\parallel}^b = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} = \frac{\frac{\eta_2}{\eta_1} \cos \theta_t - \cos \theta_i}{\frac{\eta_2}{\eta_1} \cos \theta_t + \cos \theta_i} \quad (5-24c), (15)$$

and

$$T_{\parallel}^b = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} \stackrel{\frac{1}{\eta_1}}{\stackrel{\frac{1}{\eta_1}}{=}} \frac{2\frac{\eta_2}{\eta_1} \cos \theta_i}{\frac{\eta_2}{\eta_1} \cos \theta_t + \cos \theta_i} \quad (5-24d), (16)$$

# TL Analogy

A TL analogy can be developed for H-pol UPWs incident on planar interfaces, as we did w/ L-pol UPWs.

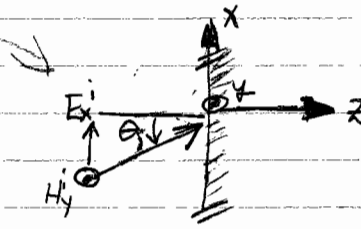
Examining  $T_{||}^b$  in (16), though, we see a problem. While  $\eta_1 \equiv \cos \theta_i$  appear together in (15) & (16), w/  $\eta_2 \equiv \cos \theta_t$  appear together in (16).

It turns out that the origin of this difficulty is with the definition of  $T_{||}^b \equiv T_{||}^b$ . We used the total  $\vec{E}$  rather than a component transverse to the interface normal.

Refer to the following figure

For this H-pol,

$$Z_{+z,1} \equiv \frac{E_x^i}{H_y^i} \quad (17)$$



and  $Z_{+z,2} \equiv \frac{E_x^t}{H_y^t}$  (18)

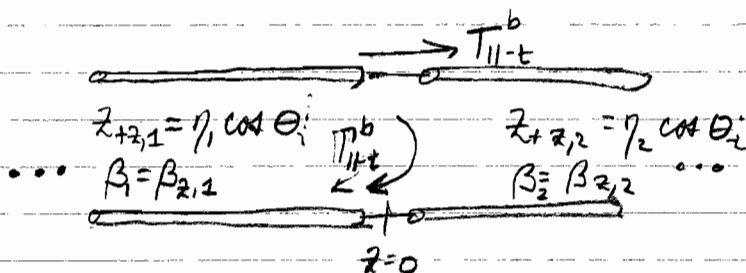
From (17) & substituting from (3) & (1) we find

$$Z_{+z,1} = \frac{\cos \theta_i E_0}{E_0 / \eta_1} = \eta_1 \cos \theta_i \quad (19)$$

& from (18) w/ (4) & (5).

$$Z_{+z,2} = \frac{\cos \theta_t T_{||}^b E_0}{T_{||}^b E_0 / \eta_2} = \eta_2 \cos \theta_t \quad (20)$$

Consequently, the TL analogous model for H-pol scattering by a planar interface is



So the reflection & transmission coeffs for the transverse electric field ( $E_x$  in this case) is

$$\Gamma_{||-t}^b \equiv \left. \frac{E_x^r}{E_x^i} \right|_{z=0} = \frac{Z_{+z,2} - Z_{+z,1}}{Z_{+z,2} + Z_{+z,1}} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} \quad (21)$$

and

$$T_{||-t}^b \equiv \left. \frac{E_x^t}{E_x^i} \right|_{z=0} = \frac{2Z_{+z,2}}{Z_{+z,2} + Z_{+z,1}} = \frac{2\eta_2 \cos \theta_t}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} \quad (22)$$

So, we can use this TL analogous model to solve  $H$ -pol problems. If we wish to know  $\Gamma_{||}^b$  &  $T_{||}^b$ , these can easily be computed from (21) & (22).

From (5) we see that  $E_x^r \Big|_{z=0} = \cos \theta_r \Gamma_{||}^b E_0 e^{-j\beta_x^r x}$  (23)  
while from (3)

$$E_x^i \Big|_{z=0} = \cos \theta_i E_0 e^{-j\beta_x^i x} \quad (24)$$

w/  $\beta_x^r = \beta_x^i \Rightarrow \theta_r = \theta_i$ , then dividing (23) & (24) gives

$$\frac{E_x^r \Big|_{z=0}}{E_x^i \Big|_{z=0}} = \Gamma_{||}^b \Rightarrow \Gamma_{||-t}^b = \Gamma_{||}^b \quad (25)$$

From (7) we see that  $E_x^t \Big|_{z=0} = \cos \theta_t T_{||}^b E_0 e^{-j\beta_x^t x}$  (26)

w/  $\beta_x^t = \beta_x^i$  then dividing (26) by (24) gives

$$\frac{E_x^t \Big|_{z=0}}{E_x^i \Big|_{z=0}} = \frac{\cos \theta_t}{\cos \theta_i} T_{||}^b \Rightarrow T_{||-t}^b = \frac{\cos \theta_t}{\cos \theta_i} T_{||}^b \quad (27)$$

— Results —

Fig 5-5

- ✓ Replot w/o mag.
- ✓ Plot  $T_{||-t}^b$
- ✓ Plot  $Z_{+z,1}$  &  $Z_{+z,2}$

# Duality & Dual Solutions

6/12

Section 7.2 in your text describes duality in Maxwell's equations. We can develop this concept by comparing similar Maxwell's equations to each other. From Table 1-4:

$\nabla \times \vec{E} = -\vec{m}_i - \vec{J}_{c,m} - j\omega\mu\vec{H}$	$\nabla \times \vec{H} = \vec{J}_i + \vec{J}_{c,e} + j\omega\epsilon\vec{E}$
$\nabla \cdot \vec{D} = \rho_{e,v}$	$\nabla \cdot \vec{B} = \rho_{m,v}$
$\nabla \cdot \vec{J}_{c,e} = -j\omega\rho_{e,v}$	$\nabla \cdot \vec{J}_{c,m} = -j\omega\rho_{m,v}$
$\vec{J}_{c,e} = \sigma_e \vec{E}$	$\vec{J}_{c,m} = \sigma_m \vec{H}$

These two sets of equations have identical forms, just the symbols are different. A simple way to construct one set of these equations from the other is:

- ✓ Replace all electric <sup>field</sup> quantities w/ dual magnetic ones
- ✓ Replace all magnetic <sup>field</sup> quantities w/ negative of dual electric ones
- ✓ interchange dual electric: magnetic material
- ✓ parameters ( $\mu \leftrightarrow \epsilon$ , which implies  $\eta \leftrightarrow \frac{1}{\eta}$ ,  $\sigma_e \leftrightarrow \sigma_m$ ).

Dual field Quantities

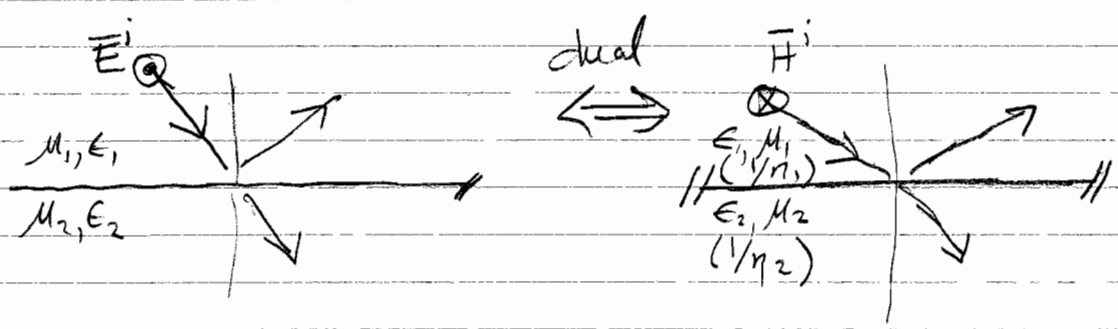
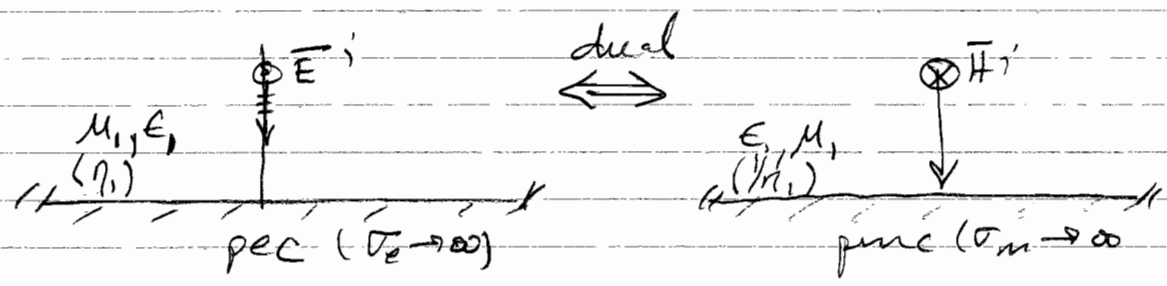
Dual Material Parameters

$\vec{E}$	$\vec{H}$	$\epsilon$	$\mu$
$\vec{D}$	$\vec{B}$	$\sigma_e$	$\sigma_m$
$\vec{J}_i$	$\vec{m}_i$		
$\vec{J}_{c,e}$	$\vec{J}_{c,m}$		
$\rho_{e,v}$	$\rho_{m,v}$		

This can be a quick check of a problem, or a simple method of constructing equations of one problem from a dual if the latter is known, but the former is not.

Additionally, if a problem is dual to another, the boundary conditions are also dual, then the full solutions for one problem will be dual to the other. (your text doesn't distinguish between duality & dual solutions.) This can be extremely useful.

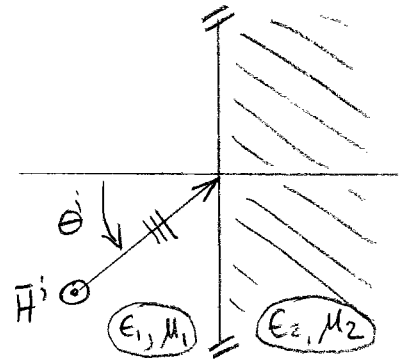
Examples of dual solutions



Example N10.1

# Parallel Polarized UPW Obliquely Incident on a Lossless Half Space - 1

EE 692 Advanced Engineering Electromagnetics  
Keith W. Whites



In[100]:=

<< Graphics`Legend`

In[101]:=

$\frac{\epsilon_1}{\epsilon_2} = \frac{1}{4}$   
 $\epsilon_{\text{lovere2}} := 1/4$   
 $\mu_{\text{lovere2}} := 1/1$  ←  $\mu_1 = \mu_2$

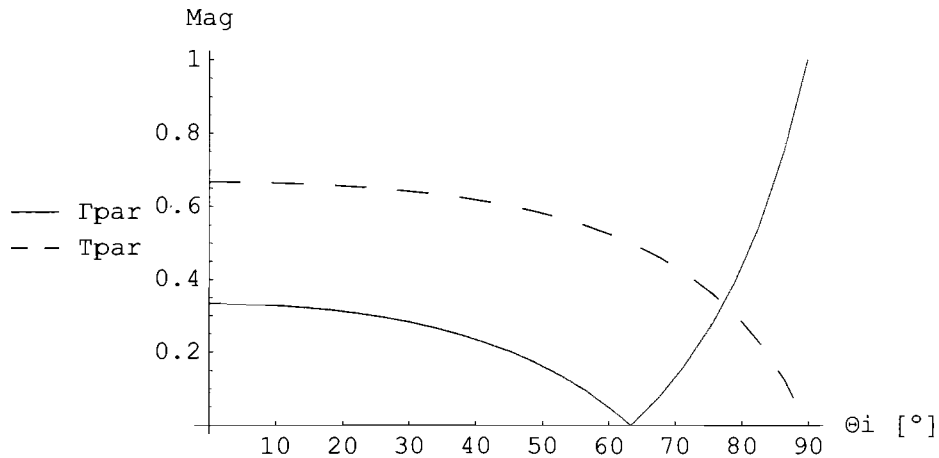
```

θt[θi_] := ArcSin[Sqrt[μlovere2 * εlovere2] * Sin[θi]]
Γpar[θi_] := (Sqrt[εlovere2 / μlovere2] * Cos[θt[θi]] - Cos[θi]) /
(Sqrt[εlovere2 / μlovere2] * Cos[θt[θi]] + Cos[θi])
Tpar[θi_] := 2 * Sqrt[εlovere2 / μlovere2] *
Cos[θi] / (Sqrt[εlovere2 / μlovere2] * Cos[θt[θi]] + Cos[θi])
    
```

In[106]:=

```

Plot[{Abs[Γpar[θi * Degree]], Abs[Tpar[θi * Degree]]}, {θi, 0, 90},
PlotRange -> Automatic, Ticks -> {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
PlotStyle -> {{RGBColor[1, 0, 0]}, {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},
AxesLabel -> {"θi [°]", "Mag"}, PlotLegend -> {"Γpar", "Tpar"},
LegendSize -> {0.5, 0.2}, LegendShadow -> None, LegendPosition -> {-1.4, -0.1}]
    
```



Out[106]=

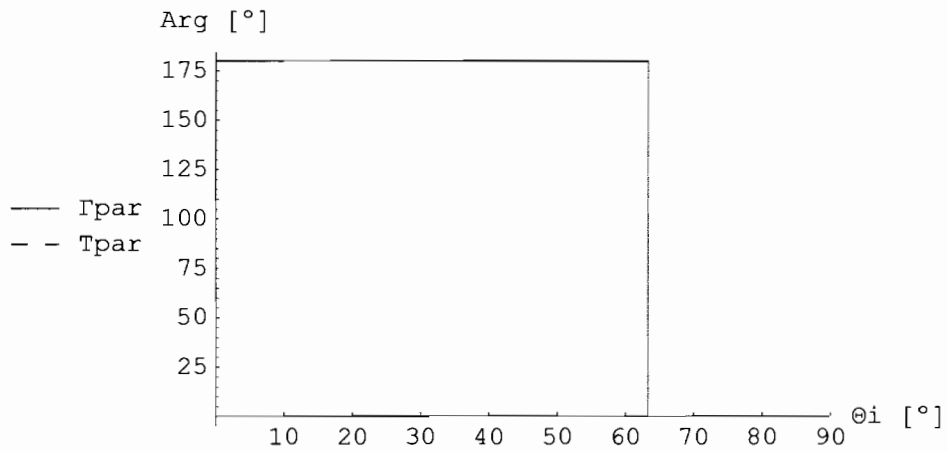
- Graphics -



## Example N10.1 (cont.)

In[107]:=

```
Plot[{Arg[GammaPar[theta * Degree]] / Degree, Arg[Tpar[theta * Degree]] / Degree}, {theta, 0, 90},  
PlotRange -> {{0, 90}, Automatic}, Ticks -> {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},  
PlotStyle -> {{RGBColor[1, 0, 0]}, {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},  
AxesLabel -> {"theta [°]", "Arg [°]"}, PlotLegend -> {"GammaPar", "Tpar"},  
LegendSize -> {0.5, 0.2}, LegendShadow -> None, LegendPosition -> {-1.4, -0.1}]
```



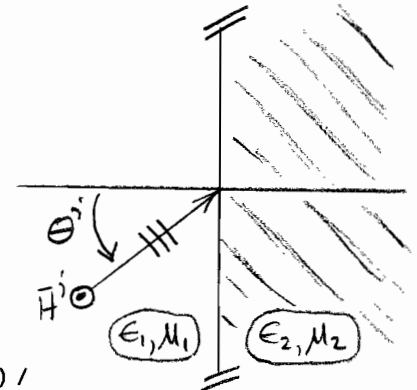
Out[107]=

• Graphics •

Example N10.2

# Parallel Polarized UPW Obliquely Incident on a Lossless Half Space - 1

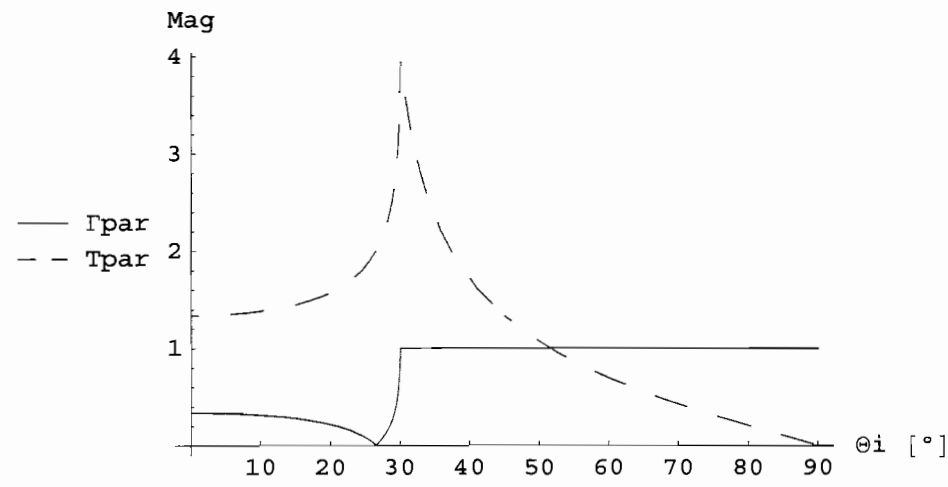
EE 692 Advanced Engineering Electromagnetics  
Keith W. Whites



```
In[10] := << Graphics`Legend`
           $\frac{\epsilon_1}{\epsilon_2} = 4$ 
In[11] :=  $\epsilon_{\text{lovere2}} := 4 / 1$ 
           $\mu_{\text{lovere2}} := 1 / 1$ 
           $\mu_1 = \mu_2$ 
```

```
 $\theta_t[\theta_i] := \text{ArcSin}[\text{Sqrt}[\mu_{\text{lovere2}} * \epsilon_{\text{lovere2}}] * \text{Sin}[\theta_i]]$ 
 $\Gamma_{\text{par}}[\theta_i] := (\text{Sqrt}[\epsilon_{\text{lovere2}} / \mu_{\text{lovere2}}] * \text{Cos}[\theta_t[\theta_i]] - \text{Cos}[\theta_i]) /$ 
                 $(\text{Sqrt}[\epsilon_{\text{lovere2}} / \mu_{\text{lovere2}}] * \text{Cos}[\theta_t[\theta_i]] + \text{Cos}[\theta_i])$ 
 $\text{Tpar}[\theta_i] := 2 * \text{Sqrt}[\epsilon_{\text{lovere2}} / \mu_{\text{lovere2}}] * \text{Cos}[\theta_i] /$ 
                 $(\text{Sqrt}[\epsilon_{\text{lovere2}} / \mu_{\text{lovere2}}] * \text{Cos}[\theta_t[\theta_i]] + \text{Cos}[\theta_i])$ 
```

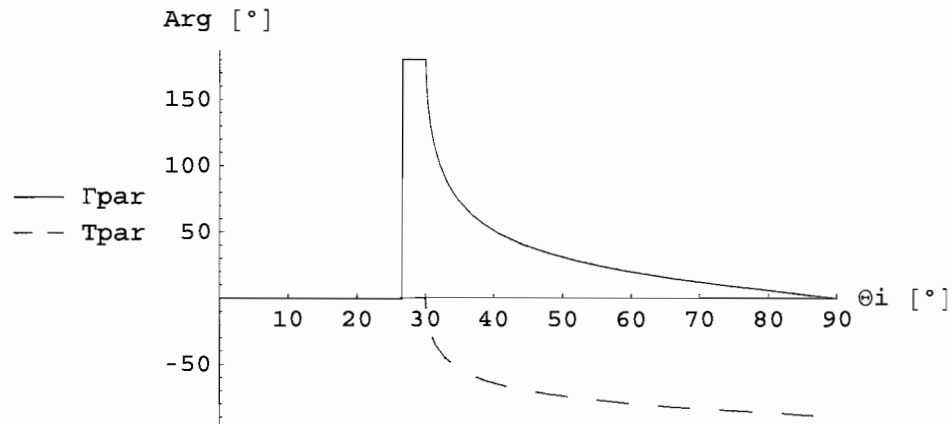
```
In[16] := Plot[{Abs[ $\Gamma_{\text{par}}[\theta_i * \text{Degree}]$ ], Abs[ $\text{Tpar}[\theta_i * \text{Degree}]$ ]}, { $\theta_i$ , 0, 90},
              PlotRange -> Automatic, Ticks -> {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
              PlotStyle -> {{RGBColor[1, 0, 0]}, {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},
              AxesLabel -> {" $\theta_i$  [°]", "Mag"}, PlotLegend -> {" $\Gamma_{\text{par}}$ ", " $\text{Tpar}$ "},
              LegendSize -> {0.5, 0.2}, LegendShadow -> None, LegendPosition -> {-1.4, -0.1}]
```



Out[16]= - Graphics -

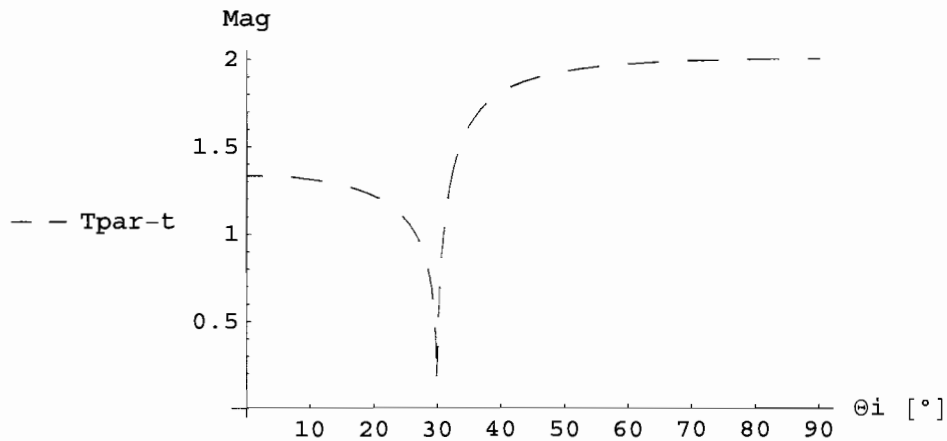
### Example N10.2 (cont.)

```
In[17]:= Plot[{Arg[Tpar[θi * Degree]] / Degree, Arg[Tpar[θi * Degree]] / Degree},
  {θi, 0, 90}, PlotRange → {{0, 90}, Automatic},
  Ticks → {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
  PlotStyle → {{RGBColor[1, 0, 0]}, {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},
  AxesLabel → {"θi [°]", "Arg [°]"}, PlotLegend → {"Tpar", "Tpar"},
  LegendSize → {0.5, 0.2}, LegendShadow → None, LegendPosition → {-1.4, -0.1}]
```



Out[17]= - Graphics -

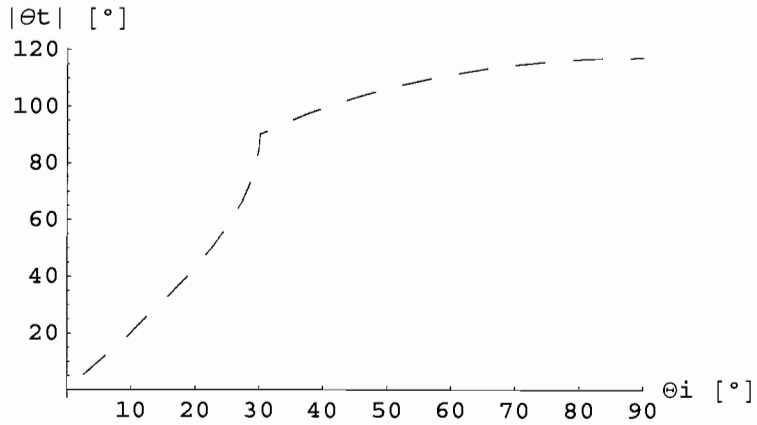
```
In[18]:= Plot[Abs[Tpar[θi * Degree] * Cos[θt[θi * Degree]] / Cos[θi * Degree]], {θi, 0, 90},
  PlotRange → Automatic, Ticks → {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},
  PlotStyle → {Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]},
  AxesLabel → {"θi [°]", "Mag"}, PlotLegend → {"Tpar-t"},
  LegendSize → {0.5, 0.2}, LegendShadow → None, LegendPosition → {-1.5, -0.1}]
```



Out[18]= - Graphics -

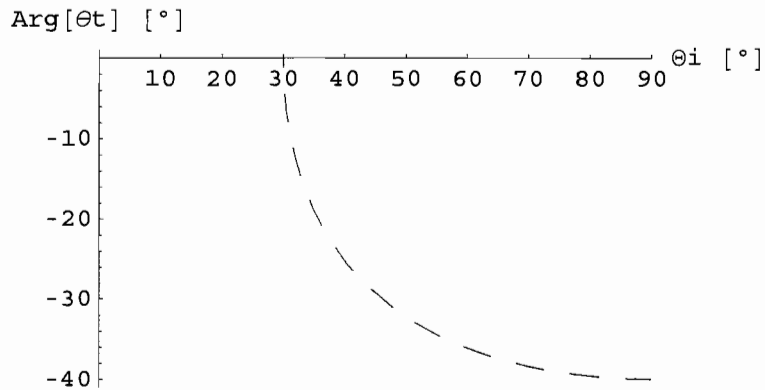
Example N10.2 (cont.)

```
In[19]:= Plot[Abs[ $\theta_t$ [ $\theta_i$ *Degree]/Degree], { $\theta_i$ , 0, 90}, PlotRange -> {{0, 90}, Automatic},  
  Ticks -> {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},  
  PlotStyle -> {{Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},  
  AxesLabel -> {" $\theta_i$  [°]", "| $\theta_t$ | [°]"}]
```



Out[19]= - Graphics -

```
Plot[Arg[ $\theta_t$ [ $\theta_i$ *Degree]/Degree]/Degree, { $\theta_i$ , 0, 90}, PlotRange -> {{0, 90}, Automatic},  
  Ticks -> {{10, 20, 30, 40, 50, 60, 70, 80, 90}, Automatic},  
  PlotStyle -> {{Dashing[{0.05, 0.05}], RGBColor[0, 0, 1]}},  
  AxesLabel -> {" $\theta_i$  [°]", "Arg[ $\theta_t$ ] [°]"}]
```



Out[20]= - Graphics -