Maxwell's Equations, Constitutive Relations, and Boundary Conditions

The focus of this course is on electromagnetic (EM) waves and their scattering by relatively simple obstacles. We aim to provide you both a mathematical foundation for this work as well as exposure to the physical principles responsible for the behavior.

There will be two principal components to this study. The first component is the propagation of EM waves themselves. We will examine the propagation of uniform plane waves (UPWs) in free space and waves guided by metallic and dielectric waveguides.

The second component is the scattering of EM waves. While this is not a course on EM scattering, per se, we will look at UPWs reflected and transmitted through layered material slabs. We will get a small taste of "true" scattering theory by looking at Physical Optics (PO) theory as applied to UPW scattering by planar strips. We will also briefly look at simple scattering of obstacles in metallic waveguides using equivalent transmission line (TL) models.

Applications of these topics include:

- ✓ Communications (satellite, submarine, DBS, etc.)
- ✓ Antenna substrates, radomes
- ✓ Microwave circuits
- ✓ Fiber optics
- ✓ LO

For this and the next lecture, we'll review basic EM theory, introduce concept of magnetic current & charge, and review interfacial boundary conditions for the EM field.

Maxwell's Equations, Magnetic Current and Charge Densities

One of the amazing things about electromagnetics is that the entire theory is adequately described in the tidy, compact set of equations called Maxwell's equations. In the time domain and in the so-called differential form, they are

Curl:

- $$\nabla \times \bar{H}(\bar{r},t) = \underbrace{\bar{J}_i(\bar{r},t)}_{\text{"impressed" or source}} + \underbrace{\bar{J}_c(\bar{r},t)}_{\text{conduction current such as } = \sigma\bar{E}} + \underbrace{\frac{\partial \bar{D}(\bar{r},t)}{\partial t}}_{\text{Displacement current}} \qquad (1\text{-}2),\ (1)$$

- $$\nabla \times \bar{E}(\bar{r},t) = \underbrace{\frac{-\partial \bar{B}(\bar{r},t)}{\partial t}}_{\text{comparing to } \frac{\partial \bar{D}}{\partial t} \text{, called a "magnetic displacement current."}} - \bar{M}_i(\bar{r},t) \qquad (1\text{-}1),\ (2)$$

In this last equation, $\bar{M}_i$ is an "impressed" (or a given source) magnetic current density [V/m²]. It is fictitious in the sense that no such physical magnetic currents have been physically observed. However, it's included to "balance" Maxwell's equations and employed as a source for certain polarizations of UPWs, and in surface equivalent problems.

<u>Divergence</u>:

- $$\nabla \cdot \bar{D}(\bar{r},t) = \underbrace{\rho_e(\bar{r},t)}_{\substack{\text{free electric charge} \\ \text{density } (C/m^3)}} \qquad (1\text{-}3), (3)$$

- $$\nabla \cdot \bar{B}(\bar{r},t) = \underbrace{\rho_m(\bar{r},t)}_{\substack{\text{by analogy, this is the} \\ \text{fictitious free magnetic charge} \\ \text{density } (Wb/m^3)}} \qquad (1\text{-}4), (4)$$

These equations have all been written in a ~~coordinate~~ free manner using vector calculus.

The magnetic current and charge concepts may appear strange at first, but they are, at the least, very useful tools. For our purposes in this course, $\bar{M}_i$ & ρ_m are strictly source quantities, though we won't be using them very often.

One point to clarify before we leave the topic of magnetic current is that $\bar{M}_i$ and the magnetization vector $\bar{M}$ are different quantities. That is, for a magnetic material

$$\bar{B} = \mu_0 \big(\underset{[A/m]}{\bar{H}} + \underset{[A/m]}{\bar{M}} \big) \qquad (2\text{-}19), (5)$$

The units of $\bar{M}$ are A/m whereas those for $\bar{M}_i$ are V/m^2. Very different quantities!

Now, in addition to these four Maxwell equations is a fifth that relates $\bar{J}_c$ & ρ_e as

$$\nabla \cdot \bar{J}_c(\bar{r},t) = \frac{-\partial \rho_e(\bar{r},t)}{\partial t} \qquad (1\text{-}6),(6)$$

which can be derived directly from (1). This equation is called the continuity equation and is an expression of conservation of charge in EM. In this case, it's electric charge.

<u>Integral Form of Maxwell's Equations</u>

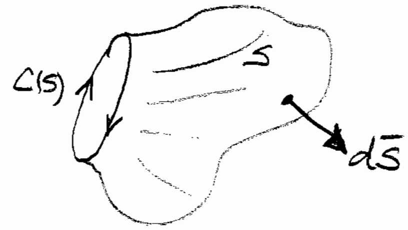
Maxwell's equations can also be written in an alternate way called the <u>integral form</u>. (The integral form can be obtained directly from the differential form, but we'll just state the result here.)

- <u>Ampere's law</u>:

$$\oint_{C(S)} \bar{H}(\bar{r},t)\cdot d\bar{l} = \int_{S(C)} \bar{J}_i(\bar{r},t)\cdot d\bar{s} + \int_{S(C)} \bar{J}_c(\bar{r},t)\cdot d\bar{s} + \underbrace{\int_{S(C)} \frac{\partial \bar{D}(\bar{r},t)}{\partial t}\cdot d\bar{s}}_{= \frac{d}{dt}\int_{S(C)} \bar{D}(\bar{r},t)\cdot d\bar{s}} \qquad (1\text{-}10),(7) \checkmark$$

All the integrals on the RHS are open surface integrals, with C and $d\bar{s}$ related by the right-hand rule :

if S & C are not varying with time.



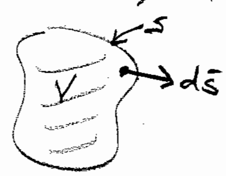
- **Faraday's Law:**

$$\oint_{C(S)} \bar{E}(\bar{r},t)\cdot d\bar{l} = \underbrace{-\int_{S(C)} \frac{\partial \bar{B}(\bar{r},t)}{\partial t}\cdot d\bar{s}}_{= -\frac{d}{dt}\int_{S(C)} \bar{B}(\bar{r},t)\cdot d\bar{s} \text{ if } S \neq f(t).} - \int_{S(C)} \bar{M}_i(\bar{r},t)\cdot d\bar{s} \qquad (1\text{-}9a),(8)$$

- **Gauss's Laws:**

$$\oint_{S(V)} \bar{D}(\bar{r},t)\cdot d\bar{s} = \int_{V(S)} \rho_e(\bar{r},t)\,dv = Q_e(t) \qquad (1\text{-}11a),(9)$$

closed surface integral; volume v enclosed by S; net *free* charge.



and

$$\oint_{S(V)} \bar{B}(\bar{r},t)\cdot d\bar{s} = \int_{V(S)} \rho_m(\bar{r},t)\,dv = Q_m(t) \qquad (1\text{-}12),(10)$$

- **Continuity equation:**

$$\oint_{S(V)} \bar{J}_c(\bar{r},t)\cdot d\bar{s} = -\int_{V(S)} \frac{\partial \rho_e(\bar{r},t)}{\partial t}\,dv \qquad (1\text{-}13),(11)$$

$$= -\frac{d}{dt} Q_e(t)$$

if $S \neq f(t)$

Constitutive Equations

There are more unknown (vector) functions in these Maxwell eqns. than there are equations $\Rightarrow$ underdetermined system. Let's look at this more closely. In the Maxwell equations we have 12 unknown scalar functions: the three components each for $\bar{H}$, $\bar{D}$, $\bar{E}$, & $\bar{B}$. How many eqns. do we have? Take a conducting material with $\bar{J}_i = \bar{M}_i = 0$ as an example. From (1) & (2):

$$\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t} \qquad (12)$$

and

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad (13)$$

What about the two equations $\nabla \cdot \bar{D} = \rho_e$ & $\nabla \cdot \bar{B} = \rho_m$? These are not independent equations from the curl eqns.

For example, taking $\nabla \cdot$ of (12)

$$\underbrace{\nabla \cdot (\nabla \times \bar{H})}_{=0 \text{ by definition}} = \nabla \cdot \bar{J}_c + \underbrace{\nabla \cdot \left(\frac{\partial \bar{D}}{\partial t} \right)}_{\text{interchange order of differentiation.}}$$

Sub. from (6):

$$\frac{\partial}{\partial t} \nabla \cdot \bar{D} - \frac{\partial \rho_e}{\partial t} = 0$$

or

$$\frac{\partial}{\partial t} \left[\nabla \cdot \bar{D} - \rho_e \right] = 0$$

This statement says that the factor in the brackets is not a fct. and equals some constant, which we take to be zero for our purposes here, giving

$$\nabla \cdot \bar{D} = \rho_e \text{ , which is (3).}$$

So what we have are six equations [(12) : (13)] and 12 unknowns.

The remaining six equations are called the constitutive equations. These equations relate $\bar{D}$ to $\bar{E}$ and $\bar{B}$ to $\bar{H}$. The specific form of these relations depend on the type of material in which the fields are located.

For example, in a so-called simple material (linear, isotropic, and homogeneous):

$$\bar{D}(\bar{r},t) = \epsilon\, \bar{E}(\bar{r},t) \qquad (1\text{-}14),(14)$$

where ϵ is the permittivity of the material [F/m], and

$$\bar{B}(\bar{r},t) = \mu\, \bar{H}(\bar{r},t) \qquad (1\text{-}15),(15)$$

where μ is the permeability of the material [H/m].

In vacuum (aka free space), $\epsilon = \epsilon_0 = 8.854 \times 10^{-12}$ F/m and $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m.

Further, if the material is conductive,

$$\bar{J}(\bar{r},t) = \sigma\, \bar{E}(\bar{r},t) \qquad (1\text{-}16),(16)$$

where σ is the electrical conductivity of the material [S/m].

Boundary Conditions

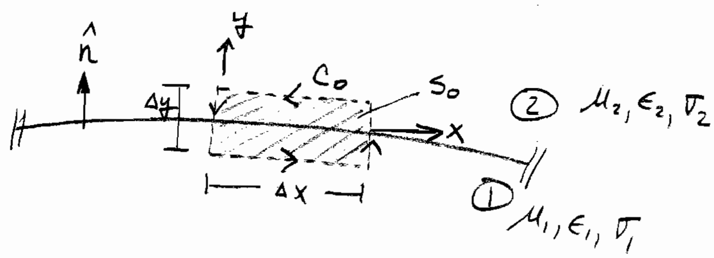
At the interface between two different EM materials, there are specific interfacial conditions the fields must obey. The discontinuity in the EM material properties (in the macroscopic model) leads to discontinuities in certain components of the fields across this boundary.

Interestingly, these boundary condition equations are derived from Maxwell's equations ~~themselves~~. You are familiar with the derivation of these boundary conditions, but we'll derive one to make a point.

Consider Faraday's law in integral form from (8) with $\overline{M}_i = 0$

$$\oint_{C_0} \overline{E} \cdot d\overline{l} = -\frac{d}{dt} \int_{S_0} \overline{B} \cdot d\overline{s} \qquad (1\text{-}25), (17)$$

at some arbitrary material interface (Fig. 1-4a)



The contour dimensions Δx and Δy are assumed small enough that $\overline{E}$ doesn't vary along each subsection of the contour. Hence, Separating the line integral in (17) into subsections gives

$$\lim_{\Delta y \to 0} \left\{ (\bar{\mathcal{E}}_1 \cdot \hat{x})\Delta x + \cancel{(\bar{\mathcal{E}}_1 \cdot \hat{y})\frac{\Delta y}{2}}^{\to 0} + \cancel{(\bar{\mathcal{E}}_2 \cdot \hat{y})\frac{\Delta y}{2}}^{\to 0} - (\bar{\mathcal{E}}_2 \cdot \hat{x})\Delta x \right.$$
$$\left. - \cancel{(\bar{\mathcal{E}}_2 \cdot \hat{y})\frac{\Delta y}{2}}^{\to 0} - \cancel{(\bar{\mathcal{E}}_1 \cdot \hat{y})\frac{\Delta y}{2}}^{\to 0} = -\frac{d}{dt}\int_{S_o} \bar{\mathcal{B}} \cdot d\bar{s} \right\} \qquad (18)$$

In the limit as $\Delta y \to 0$, S_o vanishes and the RHS of (18) tends to zero. Equation (18) then becomes

$$(\bar{\mathcal{E}}_1 \cdot \hat{x})\Delta x - (\bar{\mathcal{E}}_2 \cdot \hat{x})\Delta x = 0 \Rightarrow \underline{\underline{\mathcal{E}_{x1} = \mathcal{E}_{x2}}} \qquad (19)$$

or in general, $$\hat{n}_{21} \times (\bar{\mathcal{E}}_2 - \bar{\mathcal{E}}_1) = 0 \qquad (1\text{-}26a), (20)$$

$$(= -\mathcal{M}_s) \qquad (1\text{-}48a), (21)$$

surface magnetic current density

This boundary condition equation states that the tangential electric field is continuous across a material interface (20), or is discontinuous by an amount equal to the surface magnetic current density, if there is one (21)

This relationship is also true for static fields as well as time-varying fields. Hence, this boundary condition is sometimes called a "static" boundary condition.

Other important boundary conditions are

$$\hat{n}_{21} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s \qquad (1\text{-}39), (22)$$

$$\hat{n}_{21} \times (\bar{D}_2 - \bar{D}_1) = \rho_{es} \qquad (1\text{-}48c), (23)$$

$$\hat{n}_{21} \times (\bar{B}_2 - \bar{B}_1) = 0 \qquad (1\text{-}48d), (24)$$

$$(= \rho_{ms})$$

Good summary in Table 1-3 in the text.