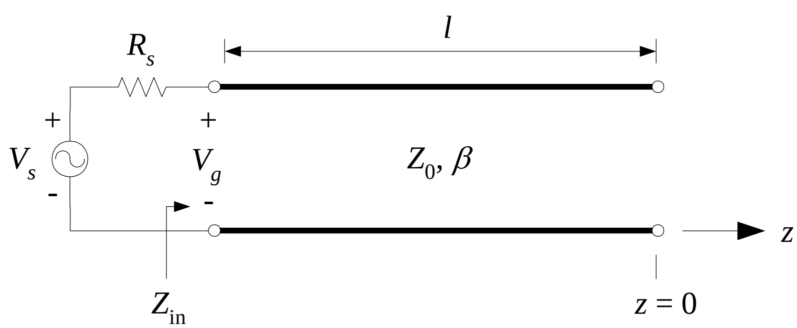
Lecture 4: TL Input Impedance, Time Average Power, Return and Insertion Losses. VSWR.

Example N4.1: Determine an expression for the voltage at the input to the TL assuming $R_s = Z_0$:



To calculate the input voltage V_g, we'll first determine the effective impedance seen at the TL input terminals seen looking towards the load at $z = 0$. This is called the **input impedance** Z_{in}.

Forming the ratio of (19) and (20) from the previous lecture gives

$$Z_{\text{in}} = \frac{V(-l)}{I(-l)} = \frac{2V_o^+ \cos(-\beta l)}{-\dfrac{j2V_o^+}{Z_0}\sin(-\beta l)} = -jZ_0 \cot(\beta l) \ [\Omega]$$

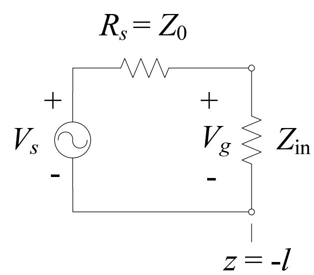
In other words, the input impedance is purely reactive

$$Z_{\text{in}} = jX_{\text{in}} \quad \text{where} \quad X_{\text{in}} = -Z_0 \cot(\beta l) \tag{2.46c}$$

A plot of this reactance is shown in Fig. 2.8c of the text.

Because of the impedance transformation properties of TLs, this input impedance Z_{in} will generally <u>not</u> equal $Z_L = \infty$. Remarkably, Z_{in} can assume any value of reactance from $-\infty$ to ∞ depending on the length l of the TL.

An equivalent circuit can now be constructed at the input to the TL by using R_s and Z_{in} as



Using voltage division,

$$V_g = \frac{Z_{\text{in}}}{Z_{\text{in}} + Z_0} V_s = \frac{-jZ_0 \cot(\beta l)}{-jZ_0 \cot(\beta l) + Z_0} V_s$$

This circuit voltage V_g is **also** the voltage on the TL at $z = -l$. That is, from (19) in the previous lecture

$$V(z = -l) = 2V_o^+ \cos(-\beta l)$$

Since $V_g = V(z = -l)$, we can equate these two voltages giving

$$2V_o^+ \cos(\beta l) = \frac{-jZ_0 \cot(\beta l)}{-jZ_0 \cot(\beta l) + Z_0} V_s$$

More often than not, expressions of this type are used to determine V_o^+ in terms of V_s, R_s, and Z_L. We'll see more on this topic in Lecture 5.

Input Impedance of a Transmission Line

In problems like the one in the last example, it is helpful to have an analytical expression for the input impedance of an arbitrarily terminated TL.

As we saw in the last lecture, the voltage and current everywhere on a homogeneous TL are

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{+j\beta z} \qquad (2.34a),(1)$$

and

$$I(z) = \frac{V_o^+}{Z_0} e^{-j\beta z} - \frac{V_o^-}{Z_0} e^{+j\beta z} \qquad (2.34b),(2)$$

We can readily construct an input impedance expression for a TL of length l by dividing (1) and (2) for some arbitrary load reflection coefficient Γ_L at $z = 0$:

$$V(-l) = V_o^+ \left(e^{+j\beta l} + \frac{V_o^-}{V_o^+} e^{-j\beta l} \right) = V_o^+ \left(e^{+j\beta l} + \Gamma_L e^{-j\beta l} \right) \qquad (3)$$

$$I(-l) = \frac{V_o^+}{Z_0} \left(e^{+j\beta l} - \frac{V_o^-}{V_o^+} e^{-j\beta l} \right) = \frac{V_o^+}{Z_0} \left(e^{+j\beta l} - \Gamma_L e^{-j\beta l} \right) \qquad (4)$$

such that

$$Z_{\text{in}} \equiv \frac{V(-l)}{I(-l)} = \frac{V_o^+ \left(e^{+j\beta l} + \Gamma_L e^{-j\beta l} \right)}{\frac{V_o^+}{Z_0} \left(e^{+j\beta l} - \Gamma_L e^{-j\beta l} \right)} = Z_0 \frac{1 + \Gamma_L e^{-j2\beta l}}{1 - \Gamma_L e^{-j2\beta l}} \qquad (2.43)$$

Substituting for Γ_L and simplifying gives

$$\boxed{Z_{\text{in}} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \quad [\Omega]} \qquad (2.44),(5)$$

This is the input impedance for a lossless TL of length l and characteristic impedance Z_0 with an arbitrary load Z_L.

Three special cases are:

1. With an open circuit load ($Z_L = \infty$), (5) yields

$$\boxed{Z_{\text{in}} = -jZ_0 \cot(\beta l) \quad [\Omega]} \qquad (2.46c),(6)$$

   as we derived in the last lecture.

2. With a short circuit load ($Z_L = 0$), (5) yields

$$\boxed{Z_{\text{in}} = jZ_0 \tan(\beta l) \quad [\Omega]} \qquad (2.45c),(7)$$

   A plot of this input reactance is shown in Fig. 2.6c.

3. With the resistive load $Z_L = Z_0$, (5) yields

$$\boxed{Z_{\text{in}} = Z_0 \quad [\Omega]}$$

   The input impedance is Z_0 regardless of the length of the TL.

All of these last three expressions should be committed to memory. You will use them often in microwave circuits.

Note that both input impedances (6) and (7) are purely reactive, which is expected since neither type can dissipate energy, assuming lossless TLs.

Time Average Power Flow on TLs

A hugely important part of microwave engineering is delivering signal power to a load. Examples include efficiently delivering power from a source to an antenna, or maximizing the power delivered from a filter to an amplifier.

Often, the "power" we are ultimately concerned with is the time average power P_{av}, expressed as

$$P_{av}(z) = \frac{1}{2}\Re e\left[V(z)I(z)^*\right] \tag{8}$$

This expression is similar to that used in circuit analysis.

Substituting $V(z)$ and $I(z)$ from (3) and (4) into (8) gives

$$P_{av}(z) = \frac{1}{2}\frac{\left|V_o^+\right|^2}{Z_0}\Re e\left[1 - \Gamma_L^* e^{-j2\beta l} + \Gamma_L e^{+j2\beta l} - \left|\Gamma_L\right|^2\right] \tag{9}$$

Notice that the second and third terms are conjugates so that

$$-\left(\Gamma_L e^{+j2\beta l}\right)^* + \Gamma_L e^{+j2\beta l} = j2\Im m\left[\Gamma_L e^{+j2\beta l}\right]$$

The real part of this sum is zero. Consequently, (9) simplifies to

$$\boxed{P_{av} = \frac{1}{2}\frac{\left|V_o^+\right|^2}{Z_0}\left(1 - \left|\Gamma_L\right|^2\right) \text{ [W]}} \tag{2.37),(10}$$

Since this power is not a function of z (true for a lossless and homogeneous TL), a z-dependence is no longer indicated for P_{av}.

It is important to reiterate that we're assuming a lossless TL throughout this analysis. These results are **not valid** for lossy TLs.

Equation (10) is very illuminating. It shows that the total time average power delivered to a load is equal to the incident time average power $\left|V_o^+\right|^2 / (2Z_0)$ minus the reflected time average power $\left|V_o^+\right|^2 \left|\Gamma\right|^2 / (2Z_0)$.

The **relative** reflected time average power from an arbitrary load on a lossless TL is the ratio of the two terms in (10) = $\left|\Gamma_L\right|^2$.

From (10) we see that if the load is entirely reactive so that $\left|\Gamma_L\right| = 1$, then $P_{\text{av}} = 0$ and no time average power is delivered to the load, as expected. For all other passive loads, $P_{\text{av}} > 0$.

The relative time average power that is not delivered to the load can be considered a "loss" since the signal from the generator was intended to be completely transported – not returned to the generator.

This return loss (RL) is defined as

$$\boxed{\text{RL} = -10\log_{10}\left(\left|\Gamma_L\right|^2\right) = -20\log_{10}\left(\left|\Gamma_L\right|\right)} \text{ dB} \qquad (2.38),(11)$$

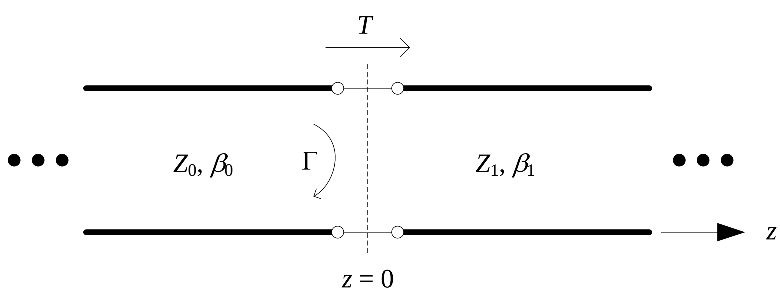
The two extremes for return loss with a passive load are:

1. A matched load where $\Gamma_L = 0$ and $\text{RL} = \infty$ dB (no reflected power), and

2. A reactive load where $\left|\Gamma_L\right| = 1$ and $\text{RL} = 0\text{ dB}$ (all power reflected).

Transmission Coefficient and Insertion Loss

Insertion loss is a term closely related to return loss. Consider a junction of two semi-infinite TLs as shown in Fig. 2.9:



We'll arbitrarily assume that a voltage wave is incident from z < 0. From (1), the total voltages in the two regions are:

$$V(z) = V_o^+\left(e^{-j\beta_0 z} + \Gamma e^{j\beta_0 z}\right) \quad z \le 0 \qquad (2.50\text{a}),(12)$$

and

$$V(z) = V_1^+ e^{-j\beta_1 z} \quad z \ge 0 \qquad (13)$$

In these expressions, V_o^+ is the complex amplitude of the incident voltage wave and V_1^+ is the complex amplitude of the transmitted voltage wave. There is no reflection on the right-hand TL so there is only the outgoing term.

We will define the voltage transmission coefficient T as

$$T = \frac{\left.V_1^+ e^{-j\beta_1 z}\right|_{z=0}}{\left.V_o^+ e^{-j\beta_0 z}\right|_{z=0}} \qquad (14)$$

so that (13) can be written as

$$V(z) = V_o^+ \frac{V_1^+}{V_o^+} e^{-j\beta_1 z} = V_o^+ T e^{-j\beta_1 z} \qquad z \geq 0 \qquad \text{(2.50b),(15)}$$

At the junction of these TLs, the two boundary conditions are that the voltage and current are each continuous across the junction. Equating voltages (12) and (15) at $z = 0$ gives

$$\boxed{T = 1 + \Gamma} \qquad \text{(2.51),(16)}$$

The reflection coefficient for this junction of two TLs is

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0} \qquad (17)$$

Substituting (17) into (16) and simplifying gives

$$\boxed{T = \frac{2Z_1}{Z_1 + Z_0}} \qquad \text{(2.51),(18)}$$

The relative time average power delivered between two "ports" in a microwave circuit is often expressed in decibels as the insertion loss, IL:

$$\boxed{\text{IL} = -10\log_{10}\left(|T|^2\right) = -20\log_{10}\left(|T|\right) \quad \text{dB}} \qquad \text{(2.52),(19)}$$

The two extremes for insertion loss in a passive circuit are:

1. A matched junction where $\Gamma = 0$, so that $T = 1$ and $\text{IL} = 0$ dB (all power transmitted), and
2. A completely reflecting junction where $\Gamma = -1$, so that $T = 0$ and $\text{IL} = \infty$ dB (no power transmitted).

Voltage Standing Wave Ratio

As we've seen, there is generally some amount of reflection of voltage and current waves from discontinuities and loads attached to a TL.

To help quantify the amount of interference that exists on a TL, we define the **voltage standing wave ratio (VSWR)** as

$$\text{VSWR} \equiv \frac{\left|V(z)\right|_{\max}}{\left|V(z)\right|_{\min}} \qquad (2.41),(20)$$

where $\left|V(z)\right|_{\max}$ and $\left|V(z)\right|_{\min}$ are the maximum and minimum voltage magnitudes, respectively, found anywhere on a *long* TL.

As shown in the text, we can determine expressions for these quantities. Specifically,

$$\left|V(z)\right|_{\max} = \left|V_o^+\right|\left(1+\left|\Gamma_L\right|\right) \qquad (2.40a),(21)$$

and

$$\left|V(z)\right|_{\min} = \left|V_o^+\right|\left(1-\left|\Gamma_L\right|\right) \qquad (2.40b),(22)$$

Substituting these into the definition of VSWR in (20) gives

$$\boxed{\text{VSWR}=\frac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|}} \qquad (2.41),(23)$$

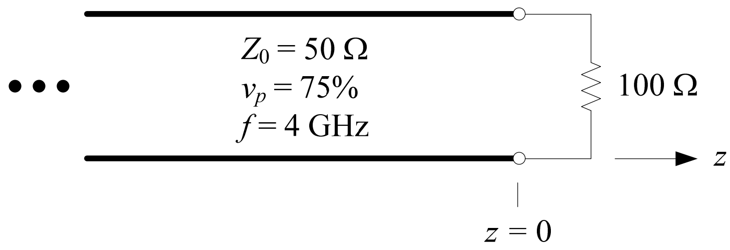
From this expression, we can definitely see that VSWR is intimately related to the amount of reflection at the load (through Γ_L) and the subsequent interference on the TL.

Special cases:

1. If $Z_L = 0$ (short-circuit load) then $\Gamma_L = -1$. Consequently, $|\Gamma_L| = 1 \quad \Rightarrow \quad \text{VSWR} = \infty$,
2. If $Z_L = \infty$ (open-circuit load) then $\Gamma_L = 1$. Consequently, $|\Gamma_L| = 1 \quad \Rightarrow \quad \text{VSWR} = \infty$,
3. If $Z_L = Z_0$ (matched load) then $\Gamma_L = 0$. Consequently, $|\Gamma_L| = 0 \quad \Rightarrow \quad \text{VSWR} = 1$.

Regardless of the load, $1 \le \text{VSWR} \le \infty$.

Example N4.2: Compute the VSWR and return loss for the TL shown below. Plot the magnitude of the phasor voltage from *z* = 0 to *z* = -7 cm. From this plot, confirm the value of VSWR that you computed earlier.



By definition, $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{100-50}{100+50} = \dfrac{1}{3}$

So, $\underline{\underline{\text{VSWR}}} = \dfrac{1+|\Gamma_L|}{1-|\Gamma_L|} = \dfrac{1+1/3}{1-1/3} \underline{\underline{= 2}}$

Trick: If Z_L & Z_0 are both real, then Γ_L is real.

Consequently, for $Z_L \geq Z_0$

$$\text{VSWR} = \frac{1+\Gamma_L}{1-\Gamma_L} \underset{\substack{\uparrow \\ \text{sub } \Gamma_L \text{ above}}}{=} \frac{1+\dfrac{Z_L-Z_0}{Z_L+Z_0}}{1-\dfrac{Z_L-Z_0}{Z_L+Z_0}} = \frac{Z_L+\cancel{Z_0}+Z_L-\cancel{Z_0}}{\cancel{Z_L}+Z_0-(\cancel{Z_L}-Z_0)}$$

$\therefore$ $\boxed{\text{VSWR} = \dfrac{Z_L}{Z_0}}$ ← If Z_L and Z_0 are real & $Z_L \geq Z_0$.

In general, $\Re\{Z_L, Z_0\}$

$$\text{VSWR} = \begin{cases} \dfrac{Z_L}{Z_0} & Z_L \geq Z_0 \\ \dfrac{Z_0}{Z_L} & Z_L < Z_0 \end{cases}$$

In this example, $Z_L = 100$ & $Z_0 = 50$

$\therefore$ $\underline{\text{VSWR}} = \dfrac{100}{50} \underline{= 2}$ as above.

Return loss $= RL = -20 \log_{10}(|\Gamma_L|)$

$\underline{\underline{RL = +9.5 \text{ dB}}}$

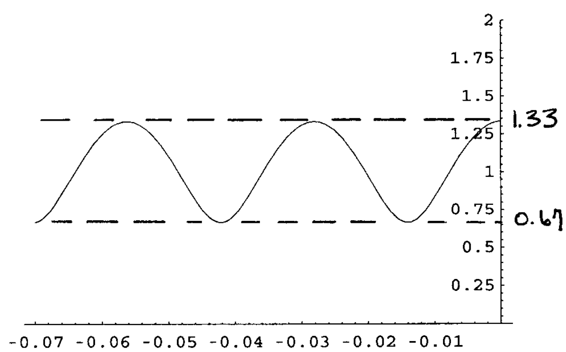
Finally, plot $|V(z)|$. $V(z) = V_0^+ \left(e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right)$

So, $|V(z)| = |V_0^+| \cdot \left| e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right|$

```
In[42]:= freq = 4 * 10^9 ;
         length = 0.07 ;
         vp = 0.75 * c0 ;
         ZL = 100 ;
         Z0 = 50 ;
         c0 = 3 * 10^8 ;
         β = 2 * Pi * freq / vp ;

         ΓL = (ZL - Z0) / (ZL + Z0)
         λ = vp / freq

Out[49]= 1/3

Out[50]= 0.05625
```

Assume $V_0^+ = 1$

```
In[51]:= Vop = 1. ;
         Voltage[z_] := Vop * (Exp[-I * β * z] + ΓL * Exp[I * β * z])
         Plot[Abs[Voltage[z]], {z, -length, 0}, PlotRange → {0, 2}]
```



$\dfrac{1.33}{0.67} \simeq 2$

checks w/ calculation above.

```
Out[53]= - Graphics -
```

Using this plot, what is VSWR if $\ell = 1$ cm?

```
In[54]:= Vmax = Abs[Vop] * (1 + Abs[ΓL])
         Vmin = Abs[Vop] * (1 - Abs[ΓL])
         Vmax / Vmin

Out[54]= 1.33333

Out[55]= 0.666667

Out[56]= 2.
```