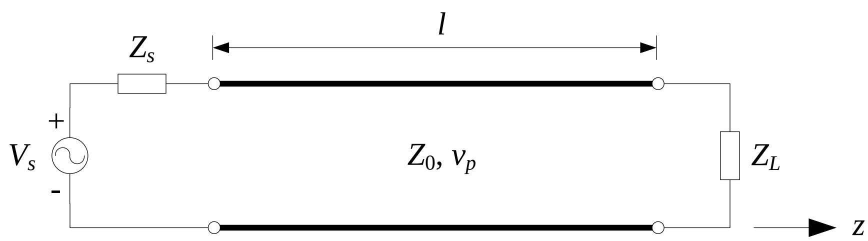
Lecture 3: Phasor Wave Solutions to the Telegrapher Equations. Termination of TLs.

We will continue our TL review by considering the steady state response of TLs to sinusoidal excitation.

Consider the following TL in the sinusoidal steady state:



We derived in the previous lecture the wave equations for the voltage and current as

$$\frac{\partial^2 v(z,t)}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 v(z,t)}{\partial t^2} \quad (1)$$

and

$$\frac{\partial^2 i(z,t)}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 i(z,t)}{\partial t^2} \quad (2)$$

For sinusoidal steady state, we will employ the phasor representation of the voltage and current as

$$v(z,t) = \Re e\left[V(z)e^{j\omega t}\right] \quad (3)$$

$$i(z,t) = \Re e\left[I(z)e^{j\omega t}\right] \quad (4)$$

where $V(z)$ and $I(z)$ are spatial phasor functions.

Substituting (3) into (1) gives

$$\frac{d^2V(z)}{dz^2} = \frac{1}{v_p^2}(j\omega)^2 V(z) = -\frac{\omega^2}{v_p^2}V(z) \qquad (5)$$

We define

$$\boxed{\beta = \omega\sqrt{LC} \quad \text{[rad/m]}} \qquad (2.12a),(6)$$

as the phase constant for reasons that will be apparent shortly. (*L* and *C* are the usual TL per-unit-length parameters.)

From (6)

$$\beta^2 = \omega^2 LC = \frac{\omega^2}{v_p^2}$$

Substituting this into (5) gives

$$\boxed{\frac{d^2V(z)}{dz^2} + \beta^2 V(z) = 0} \qquad (7)$$

Similarly, from (4) and (2) we can derive

$$\boxed{\frac{d^2I(z)}{dz^2} + \beta^2 I(z) = 0} \qquad (8)$$

Equations (7) and (8) are the wave equations for *V* and *I* in the frequency domain (i.e., the phasor domain).

The solutions to these two second-order ordinary differential equations are

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{+j\beta z} \qquad (2.14a),(9)$$

and

$$I(z) = I_o^+ e^{-j\beta z} + I_o^- e^{+j\beta z} \qquad (10)$$

$V_o^+, V_o^-, I_o^+, I_o^-$ are complex constants.

We can confirm the correctness of these two solutions by direct substitution into (7) and (8). For example, substituting $V_o^+ e^{-j\beta z}$ from (9) into (7) gives

$$V_o^+ \left(-j\beta\right)^2 e^{-j\beta z} + \beta^2 V\left(z\right) \stackrel{?}{=} 0$$

or

$$-\beta^2 V_o^+ e^{-j\beta z} + \beta^2 V_o^+ e^{-j\beta z} \stackrel{?}{=} 0$$

which is indeed true. Therefore, $V_o^+ e^{-j\beta z}$ in (9) is a valid solution to (7).

The constants I_o^+ and I_o^- in (10) can be expressed in terms of V_o^+ and V_o^-. In particular, it can be shown that

$$I_o^+ = \frac{V_o^+}{Z_0} \tag{11}$$

and

$$I_o^- = -\frac{V_o^-}{Z_0} \tag{12}$$

If we substitute (11) and (12) into (10) we find that

$$\boxed{I\left(z\right) = \frac{V_o^+}{Z_0} e^{-j\beta z} - \frac{V_o^-}{Z_0} e^{+j\beta z}} \qquad \text{(2.14b),(13)}$$

and

$$\boxed{V\left(z\right) = V_o^+ e^{-j\beta z} + V_o^- e^{+j\beta z}} \qquad \text{(2.14a),(14)}$$

Both of these equations should be committed to memory. They are the general form of phasor voltages and currents on transmission lines.

The first terms in (13) and (14) are the phasor representation of waves propagating in the +z direction along the TL. The second terms in both equations represent waves propagating in the -z direction.

Discussion

- As stated above, the first terms in (13) and (14) are the phasor representation of waves traveling in the +z direction. To see this, convert the first term in (14) to the time domain:

$$v(z,t) = \Re e\left[V_o^+ e^{-j\beta z} e^{j\omega t}\right] = \Re e\left[\left|V_o^+\right| e^{j\phi^+} e^{j(\omega t - \beta z)}\right]$$

$$= \left|V_o^+\right| \cos\left(\omega t - \beta z + \phi^+\right) = \left|V_o^+\right| \cos\left[\omega\left(t - \frac{\beta}{\omega} z\right) + \phi^+\right]$$

$$= \left|V_o^+\right| \cos\left[\omega\left(t - \frac{z}{v_p}\right) + \phi^+\right]$$

  We can clearly see in this last result that we have a function of time with argument $t - z/v_p$. From our previous discussions with TLs we recognize that this is a wave that is propagating in the +z direction with speed v_p.

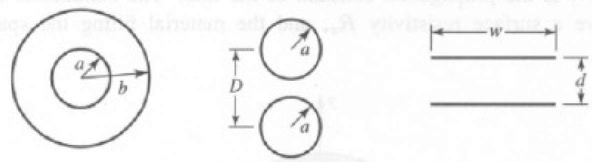
- Similarly, we can show that $V_o^- e^{+j\beta z}$ (and $I_o^- e^{+j\beta z}$) are waves propagating in the -z direction.

Generality of TL Theory

It was mentioned in the last lecture that transmission lines could be used to model the voltage and current waves on any structure supporting only TEM waves.

What changes from structure to structure are the values for L, C, R, and G, as shown below in Table 2.1 for three common TEM structures:

TABLE 2.1 Transmission Line Parameters for Some Common Lines

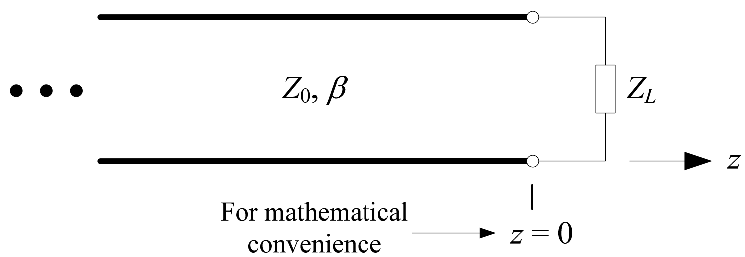


	COAX	TWO-WIRE	PARALLEL PLATE
L	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$	$\frac{\mu d}{w}$
C	$\frac{2\pi\epsilon'}{\ln b/a}$	$\frac{\pi\epsilon'}{\cosh^{-1}(D/2a)}$	$\frac{\epsilon' w}{d}$
R	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{R_s}{\pi a}$	$\frac{2R_s}{w}$
G	$\frac{2\pi\omega\epsilon''}{\ln b/a}$	$\frac{\pi\omega\epsilon''}{\cosh^{-1}(D/2a)}$	$\frac{\omega\epsilon'' w}{d}$

Consequently, the values of Z_0, v_p, β (and α) generally all change from one TL to another. The numerical values can also be changed within a type of TL by varying the dimensions and construction materials.

Termination of Transmission Lines

We will now consider the **termination** of TLs that are excited by sinusoidal steady state sources.



Adding terminations produces reflections so that the total voltage and current anywhere on the TL are sums of forward and reverse propagating waves. From (13) and (14), the voltage and current on the TL will have the form

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{+j\beta z} \tag{15}$$

and

$$I(z) = \frac{V_o^+}{Z_0} e^{-j\beta z} - \frac{V_o^-}{Z_0} e^{+j\beta z} \tag{16}$$

The "lumped load" Z_L that terminates the TL is considered a boundary condition for the voltage and current in (15) and (16):

$$V(z=0) = I(z=0) Z_L \tag{17}$$

Therefore, we can solve for V_o^- in terms of V_o^+ by applying this boundary condition as:

- from (15): $V(z=0) = V_o^+ + V_o^-$
- from (16): $I(z=0) = \frac{1}{Z_0}\left(V_o^+ - V_o^-\right)$

Forming the ratio of these quantities gives

$$\frac{V(0)}{I(0)} = \frac{V_o^+ + V_o^-}{V_o^+ - V_o^-} Z_0 = Z_L$$

Solving for V_o^- / V_o^+, and defining this ratio as the voltage reflection coefficient at the load ($z = 0$), we find

$$\Gamma_L \equiv \left. \frac{V_o^-}{V_o^+} \right|_{z=0} = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad (2.35),(18)$$

Note that, in general, Γ_L is a complex number since Z_L is complex.

Example N3.1: For an open-circuit load on the TL shown above, compute the load reflection coefficient and sketch the voltage and current magnitude on the TL.

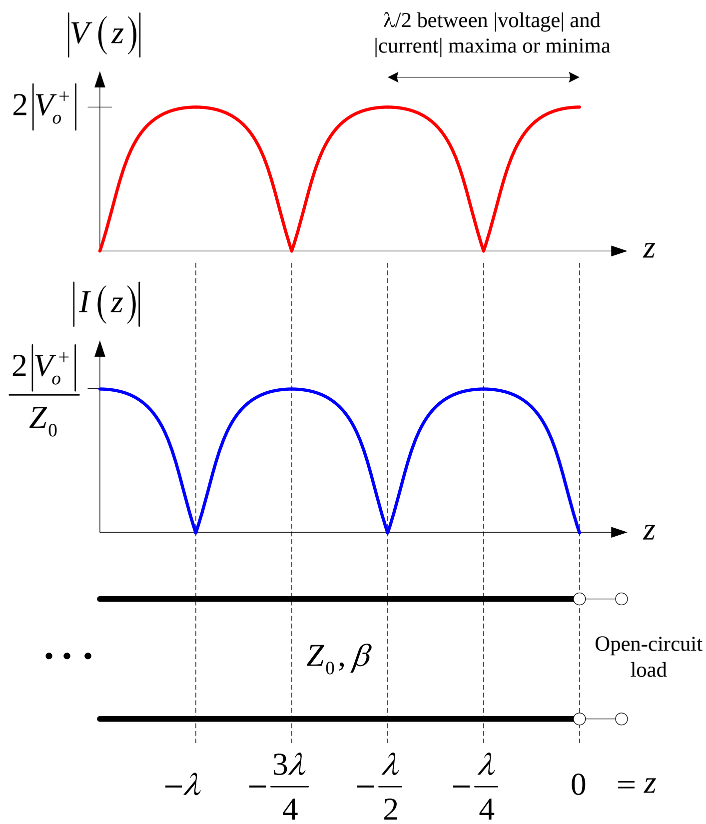
For an open circuit load $Z_L = \infty$ (i.e., an extremely large impedance), so that from (18) $\Gamma_L = +1$. With this value of Γ_L, then from (15) and (16) the solutions for $V(z)$ and $I(z)$ are

$$V(z) = V_o^+ \left(e^{-j\beta z} + \underbrace{\frac{V_o^-}{V_o^+}}_{=\Gamma_L} e^{j\beta z} \right) = 2V_o^+ \cos(\beta z) \qquad (2.46a),(19)$$

and $$I(z) = \frac{V_o^+}{Z_0} \left(e^{-j\beta z} - \underbrace{\frac{V_o^-}{V_o^+}}_{=\Gamma_L} e^{j\beta z} \right) = -\frac{j2V_o^+}{Z_0} \sin(\beta z) \qquad (2.46b),(20)$$

These two equations (19) and (20) are **not** traveling waves. So, where has the traveling wave behavior in $V(z)$ and $I(z)$ gone? The **interference** between the incident and reflected waves produces **standing waves**, such as these.

$|V(z)|$ and $|I(z)|$ are shown here for the open-circuit load:



Phasor voltage and current magnitudes vary noticeably along TLs provided the TL length is greater than about 0.05 λ or so.

Remember, though, that we are plotting the **magnitude** of phasor voltages and currents. The voltage and current oscillate as functions of time with amplitudes equal to $\left|V(z)\right|$ and $\left|I(z)\right|$, respectively, at each point along the TL.