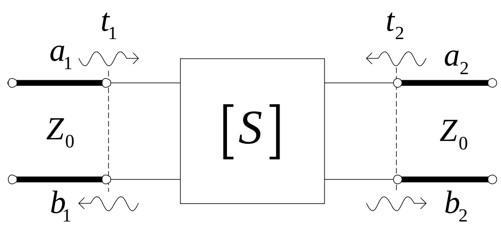
Lecture 21: Signal Flow Graphs.

Consider the following two-port network (Fig 4.14a):



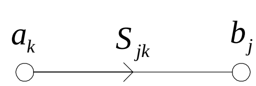
(Fig. 1)

A signal flow graph is a diagram depicting certain relationships between signals in a **linear** network. It can also be used to solve for ratios of these signals.

Signal flow graphs are used in control systems, power systems, and other fields besides microwave engineering.

Key elements of a signal flow graph are:

1. Nodes represent the system variables,
2. Branches represent paths for signal flow.
3. A signal S_{jk} traveling along a branch between nodes a_k and b_j is multiplied by the gain of that branch:



That is, $$b_j = S_{jk} a_k$$

4. Signals travel along branches only in the direction of the arrows.

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This restriction exists so that a branch from a_k to b_j denotes a proportional dependence of b_j on a_k, but not the reverse.

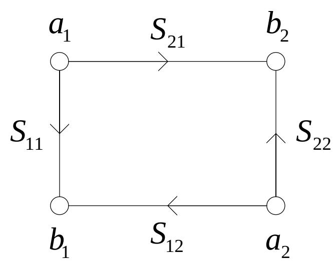
5. Multiple branches into a node represent a summation of signals.

For example, referring to the two-port above, where

$$b_1 = S_{11}a_1 + S_{12}a_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2$$

the signal flow graph is (Fig 4.14b):



(Fig. 2)

The signals a_1, a_2, b_1, and b_2 are denoted by nodes whereas certain interactions between the signals are denoted by the branches and the transmission factors (S parameters) of those branches.

Construction of Signal Flow Graphs

Let's take a more careful look at the construction of the signal flow graph (SFG) in Fig. 2. Starting with the definition of the matrix description of the two port in Fig. 1

$$[b] = [S] \cdot [a] \quad (1)$$

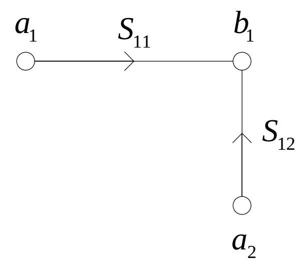
or

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (2)$$

The first equation in (2) is

$$b_1 = S_{11} a_1 + S_{12} a_2 \quad (3)$$

Using items 3 and 5 listed above, the SFG representation for (3) is

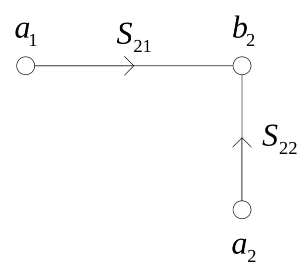


(Fig. 3)

The second equation in (2) is

$$b_2 = S_{21} a_1 + S_{22} a_2 \quad (4)$$

Using items 3 and 5 listed above, the SFG representation for (4) is

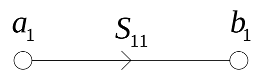


(Fig. 4)

Combining these two SFGs in Figs. 3 and 4 gives the overall SFG show earlier in Fig. 2.

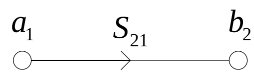
If we wish to construct just a single branch of this signal flow graph we only need to match the transmitted port. For example, if $a_2 = 0$, we see from (3) and Fig. 3 that

$$b_1 = S_{11} a_1 \qquad \Rightarrow$$



while from (4) and Fig. 4

$$b_2 = S_{21} a_1 \qquad \Rightarrow$$



Solving Signal Flow Graphs

Signal flow graphs can form an intuitive picture of the signal flow in a network. As an application, we will develop SFGs in the next lecture to help us calibrate out systematic errors present when we make measurements with a VNA.

Another useful characteristic is that we can solve for ratios of signals directly from a SFG using a simple graphical algebra.

There are four rules that form the algebra of SFGs:

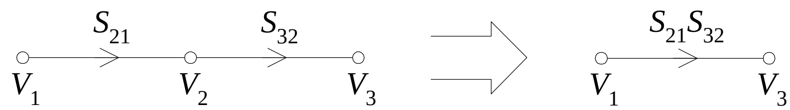
1. **Series Rule**. Given the two proportional relations

$$V_2 = S_{21} V_1 \quad \text{and} \quad V_3 = S_{32} V_2$$

then

$$V_3 = \left(S_{32} S_{21} \right) V_1 \qquad (4.75),(5)$$

In a SFG, this is represented as (Fig. 4.16a):

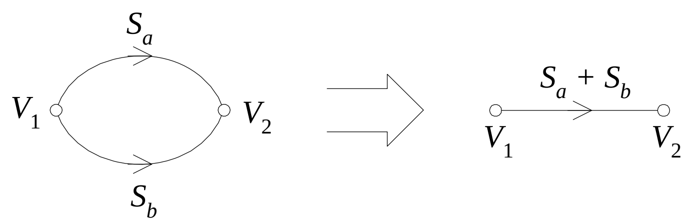


In other words, two series paths are equivalent to a single path with a transmission factor equal to a product of the two original transmission factors.

2. **Parallel Rule**. Consider the relation:

$$V_2 = S_a V_1 + S_b V_1 = (S_a + S_b) V_1 \qquad (4.76),(6)$$

In a SFG, this is represented as (Fig 4.16b):



In other words, two parallel paths are equivalent to a single path with a transmission factor equal to the sum of the original transmission coefficients.

3. **Self-Loop Rule**. Consider the relations

$$V_2 = S_{21} V_1 + S_{22} V_2 \qquad (4.77a),(7)$$

and

$$V_3 = S_{32} V_2 \qquad (4.77b),(8)$$

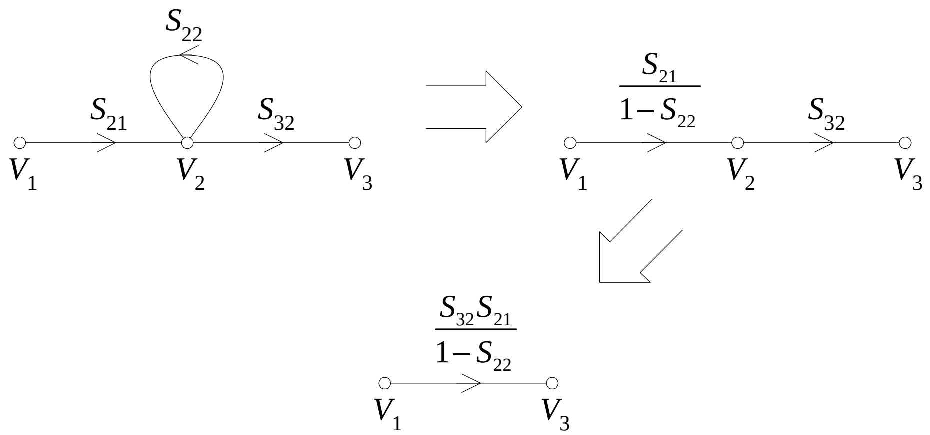
We will choose to eliminate V_2 from these two equations. From (7)

$$V_2(1-S_{22}) = S_{21}V_1 \quad \Rightarrow \quad V_2 = \frac{S_{21}}{1-S_{22}}V_1$$

Substituting this into (8) gives

$$V_3 = \frac{S_{32}S_{21}}{1-S_{22}}V_1 \qquad (4.78),(9)$$

In a SFG, this is represented as (Fig 4.16c):



In other words, a feedback loop may be eliminated by dividing the input transmission factor by one minus the transmission factor around the loop.

4. **Splitting Rule**. Consider the relationships:

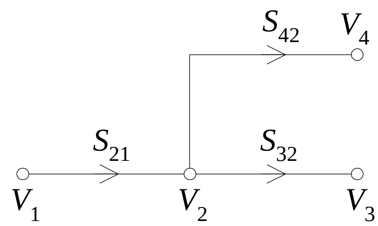
$$V_4 = S_{42}V_2 \qquad (10)$$

$$V_2 = S_{21}V_1 \qquad (11)$$

and

$$V_3 = S_{32}V_2 \qquad (12)$$

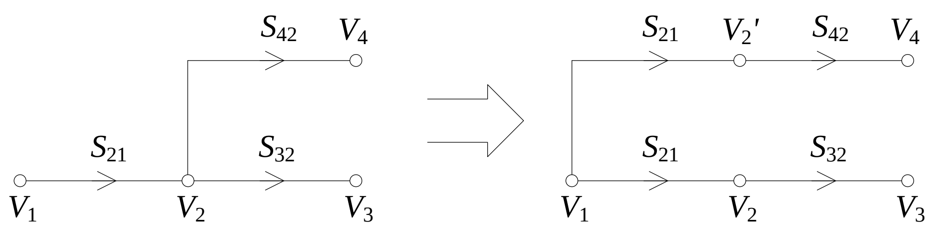
The SFG is



From (10) and (11) we find that $V_4 = S_{42}S_{21}V_1$. Hence, if we use the Series Rule “in reverse” we can define:

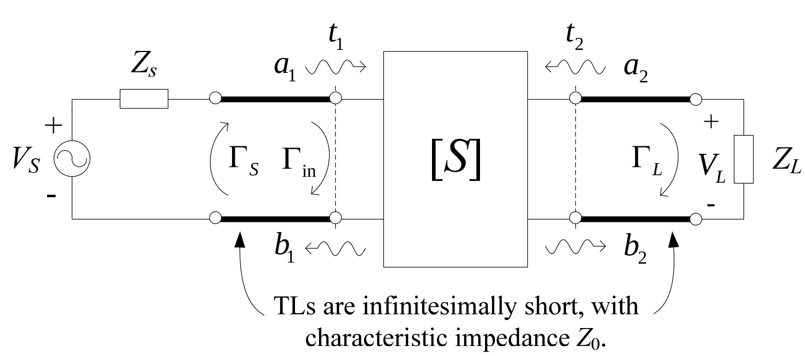
$$V_4 = S_{42}V_2' \quad \text{and} \quad V_2' = S_{21}V_1$$

In a SFG, this is represented as (Fig 4.16d):



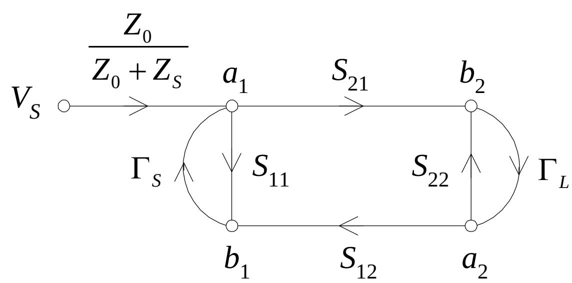
In other words, a node can be split such that the product of transmission factors from input to output is unchanged.

Example N21.1. Construct a signal flow graph for the network shown below. Determine Γ_{in} and V_L using only SFG algebra.



(Fig. 5)

The signal flow graph is:



Notice the arrow directions for Γ_S and Γ_L. These are the correct orientations since

$$\Gamma_S = \frac{a_1}{b_1} \quad \Rightarrow \quad a_1 = \Gamma_S b_1$$

and

$$\Gamma_L = \frac{a_2}{b_2} \quad \Rightarrow \quad a_2 = \Gamma_L b_2$$

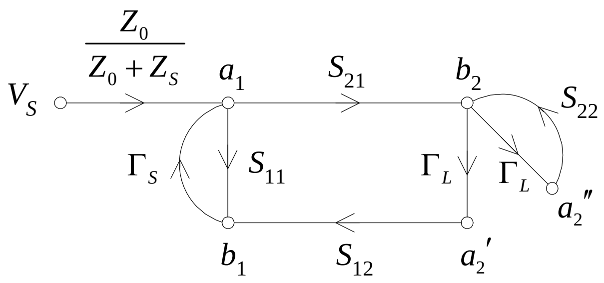
Also notice the branch relating V_S and a_1 in Fig. 5. As mentioned near the beginning of this lecture, a single branch relationship between two nodes can be found by matching the transmitted port. In this case, we match the TL connected to the source V_S giving only a forward propagating wave with a voltage amplitude found from voltage division to be

$$a_1 = \frac{Z_0}{Z_0 + Z_S} V_S \tag{13}$$

We will systematically apply the four rules above to reduce this diagram to a form that will directly allow us to determine both Γ_{in} and V_L.

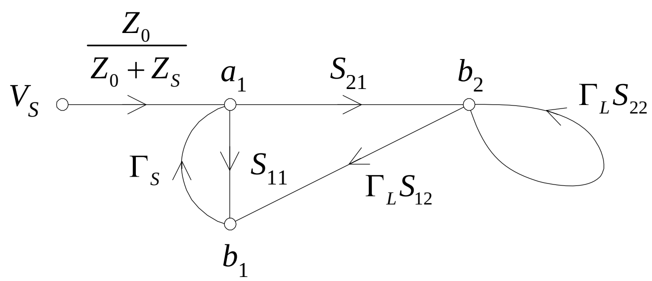
Starting this solution process is probably the most difficult part. The rest is fairly systematic.

Step 1. Start by splitting node a_2 using Rule 4:

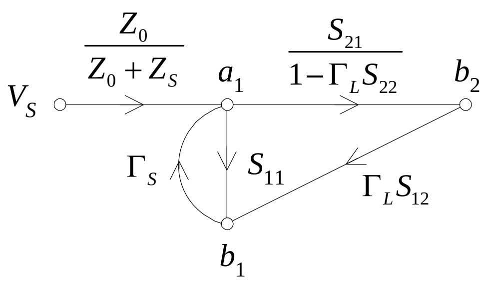


Why split the Γ_L branch and not the S_{22} branch? Because the Γ_L arrow is in the same direction as S_{12}.

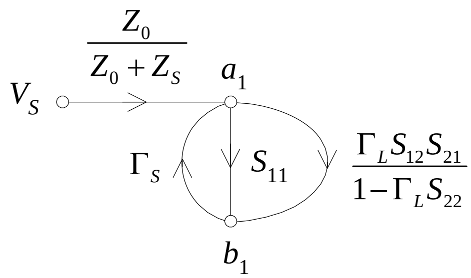
Step 2. Eliminate nodes a_2' and a_2'' using Rule 1 twice:



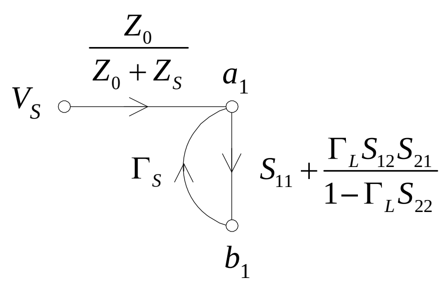
Step 3. Eliminate the self loop at b_2 using Rule 3:



Step 4. Eliminate node b_2 using Rule 1:



Step 5. Apply Rule 2:



From this last diagram we can directly solve for Γ_{in}:

$$\boxed{\Gamma_{\text{in}} \equiv \frac{b_1}{a_1} = S_{11} + \frac{\Gamma_L S_{12} S_{21}}{1-\Gamma_L S_{22}}} \tag{14}$$

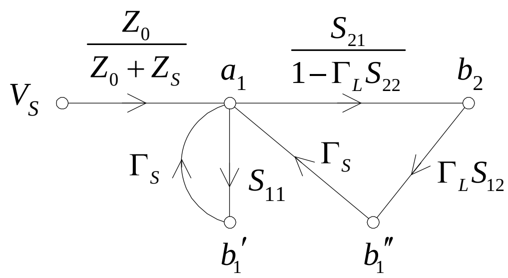
Next, we will determine the load voltage V_L. The voltage on this TL can be expressed as $V_2(z)=b_2\left(e^{-j\beta z}+\Gamma_L e^{j\beta z}\right)$. At the terminal plane $z=0$, then

$$V_2(0)=V_L=b_2(1+\Gamma_L) \tag{15}$$

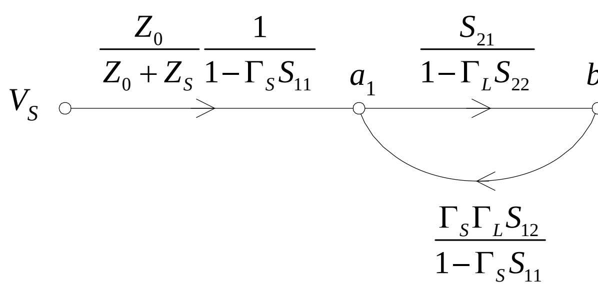
In this expression, $V_2(0)=V_L$ because the TL is very short.

So, we see from (15) that to find V_L we need to determine b_2. In Step 4, however, we eliminated that node. Let's start again from Step 3, but now split node b_1:

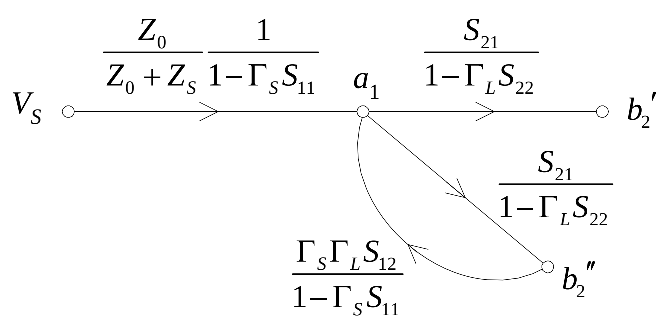
Step 4′. Split node b_1 using Rule 4:



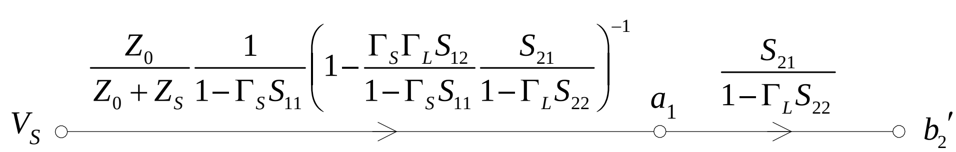
Step 5′. Next, we can use Rule 1, then Rule 3 (Self-Loop Rule) to all branches feeding node a_1:



Step 6′. Split node b_2 using Rule 4:



Step 7′. Apply Rule 3 one last time to remove the self loop:



Step 8′. Using the Series Rule, we can now find b_2 as:

$$b_2 = \frac{Z_0}{Z_0 + Z_S} \frac{1}{1 - \Gamma_S S_{11}} \left[\frac{(1 - \Gamma_S S_{11})(1 - \Gamma_L S_{22}) - \Gamma_S \Gamma_L S_{12} S_{21}}{(1 - \Gamma_S S_{11})(1 - \Gamma_L S_{22})} \right]^{-1} \frac{S_{21}}{1 - \Gamma_L S_{22}} V_S$$

$$= \frac{Z_0 S_{21} V_S (1 - \Gamma_S S_{11})(1 - \Gamma_L S_{22})}{(Z_0 + Z_S)(1 - \Gamma_S S_{11})[(1 - \Gamma_S S_{11})(1 - \Gamma_L S_{22}) - \Gamma_S \Gamma_L S_{12} S_{21}](1 - \Gamma_L S_{22})}$$

or

$$b_2 = \frac{Z_0 S_{21} V_S}{(Z_0 + Z_S)\left[(1 - \Gamma_S S_{11})(1 - \Gamma_L S_{22}) - \Gamma_S \Gamma_L S_{12} S_{21}\right]}$$

Using this b_2, we can determine V_L from (15) to be

$$V_L = \frac{Z_0 S_{21} (1 + \Gamma_L)}{(Z_0 + Z_S)\left[(1 - \Gamma_S S_{11})(1 - \Gamma_L S_{22}) - \Gamma_S \Gamma_L S_{12} S_{21}\right]} V_S$$

Reference

R. E. Collin, *Foundations for Microwave Engineering*. New York: McGraw-Hill, second ed., 1992.