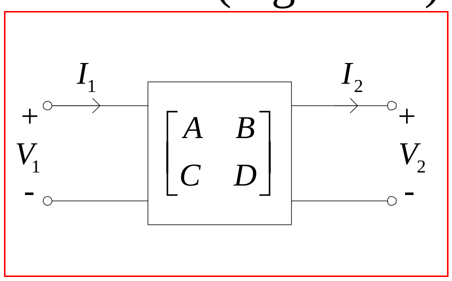
Lecture 20: Transmission (*ABCD*) Matrix.

Concerning the equivalent port representations of networks we've seen in this course:

1. Z parameters are useful for series connected networks,
2. Y parameters are useful for parallel connected networks,
3. S parameters are useful for describing interactions of voltage and current waves with a network.

There is another set of network parameters particularly suited for cascading two-port networks. This set is called the *ABCD* matrix or, equivalently, the transmission matrix.

Consider this two-port network (Fig. 4.11a):



Unlike in the definition used for Z and Y parameters, notice that I_2 is directed away from the port. This is an important point and we'll discover the reason for it shortly.

The *ABCD* matrix is defined as

$$\boxed{\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}} \qquad (4.69),(1)$$

It is easy to show that

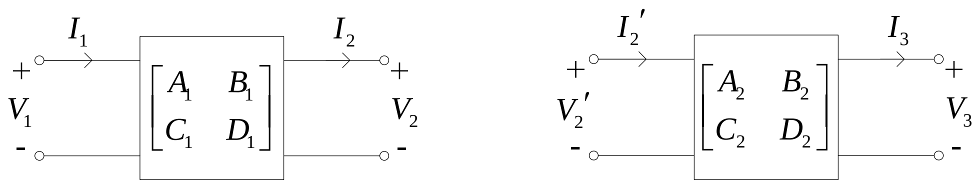
$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}, \quad B = \left.\frac{V_1}{I_2}\right|_{V_2=0}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}, \quad D = \left.\frac{I_1}{I_2}\right|_{V_2=0}$$

Note that not all of these parameters have the same units.

The usefulness of the *ABCD* matrix is that cascaded two-port networks can be characterized by simply multiplying their *ABCD* matrices. Nice!

To see this, consider the following two-port networks:



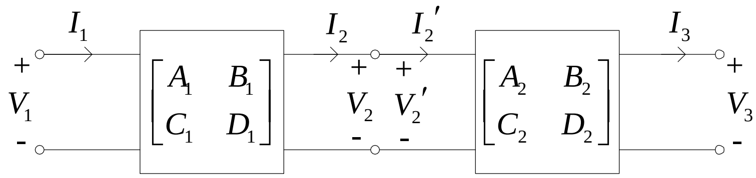
In matrix form

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \qquad \text{(4.70a),(2)}$$

and

$$\begin{bmatrix} V_2' \\ I_2' \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \cdot \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \qquad (3)$$

When these two-ports are cascaded,



it is apparent that $V_2' = V_2$ and $I_2' = I_2$. (The latter is the reason for assuming I_2 out of the port.) Consequently, substituting (3) into (2) yields

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \cdot \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \qquad (4.71),(4)$$

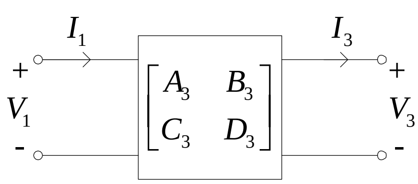
We can consider the matrix-matrix product in this equation as describing the cascade of the two networks. That is, let

$$\begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \qquad (5)$$

so that

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} \cdot \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \qquad (6)$$

where



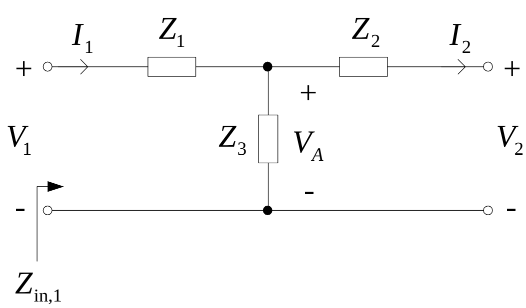
In other words, a cascaded connection of two-port networks is equivalent to a single two-port network containing a product of the *ABCD* matrices.

It is important to note that the order of matrix multiplication must be the same as the order in which the two ports are arranged in the circuit from signal input to output. Matrix multiplication is not commutative, in general. That is, $[A]\cdot[B]\neq[B]\cdot[A]$.

Text example 4.6 shows the derivation of the *ABCD* parameters for a series (i.e., "floating") impedance, which is the first entry in Table 4.1 on p. 190 of the text.

In your homework, you'll derive the *ABCD* parameters for the next three entries in the table. In the following example, we'll derive the last entry in this table.

Example N20.1 Derive the *ABCD* parameters for the *T* network:



Recall from (1) that by definition

$$V_1 = AV_2 + BI_2 \quad \text{and} \quad I_1 = CV_2 + DI_2$$

- To determine *A*: $\displaystyle A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$

we need to open-circuit port 2 so that $I_2 = 0$. Hence,

$$V_A = \frac{Z_3}{Z_1 + Z_3} V_1 = V_2$$

which yields,
$$\boxed{A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = 1 + \frac{Z_1}{Z_3}}$$

- To determine *B*:
$$B = \left.\frac{V_1}{I_2}\right|_{V_2=0}$$

we need to short-circuit port 2 so that $V_2 = 0$. Then, using current division:

$$I_2 = \frac{Z_3}{Z_2 + Z_3} I_1$$

Substituting this into the expression for *B* above we find

$$B = \left.\underbrace{\frac{V_1}{I_1}}_{=Z_{\text{in},1}|_{V_2=0}} \cdot \left(1 + \frac{Z_2}{Z_3}\right)\right|_{V_2=0} = \left(Z_1 + Z_2 \| Z_3\right)\left(1 + \frac{Z_2}{Z_3}\right)$$

$$= Z_1 + \frac{Z_1 Z_2}{Z_3} + Z_2 \| Z_3 \left(1 + \frac{Z_2}{Z_3}\right)$$

$$= Z_1 + \frac{Z_1 Z_2}{Z_3} + \frac{Z_2 Z_3}{Z_2 + Z_3}\frac{Z_3 + Z_2}{Z_3}$$

Therefore,
$$\boxed{B = Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3}}$$

- To determine C: $$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

  we need to open-circuit port 2, from which we find
  $$V_A = I_1 Z_3 = V_2$$
  Therefore, $$\boxed{C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{1}{Z_3}}$$

- To determine D: $$D = \left.\frac{I_1}{I_2}\right|_{V_2=0}$$

  we need to short-circuit port 2. Using current division, as above,
  $$I_2 = \frac{Z_3}{Z_2 + Z_3} I_1$$
  Therefore, $$\boxed{D = \left.\frac{I_1}{I_2}\right|_{V_2=0} = 1 + \frac{Z_2}{Z_3}}$$

These *ABCD* parameters agree with those listed in the last entry of Table 4.1.

Properties of *ABCD* parameters

As shown on p. 191 of the text, the *ABCD* parameters can be expressed in terms of the Z parameters. (Actually, there are interrelationships between all the network parameters, which are conveniently listed in Table 4.2 on p. 192.)

From this relationship, we can show that for a reciprocal network

$$\text{Det}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = 1 \quad \text{or} \quad \boxed{AD - BC = 1}$$

If the network is lossless, there are no really outstanding features of the *ABCD* matrix. Rather, using the relationship to the *Z* parameters we can see that if the network is lossless, then

- From (4.73a): $A = \frac{Z_{11}}{Z_{21}} \Rightarrow A$ real
- From (4.73b): $B = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}} \Rightarrow B$ imaginary
- From (4.73c): $C = \frac{1}{Z_{21}} \Rightarrow C$ imaginary
- From (4.73d): $D = \frac{Z_{22}}{Z_{21}} \Rightarrow D$ real

In other words, the diagonal elements are real while the off-diagonal elements are imaginary for an *ABCD* matrix representation of a lossless network.