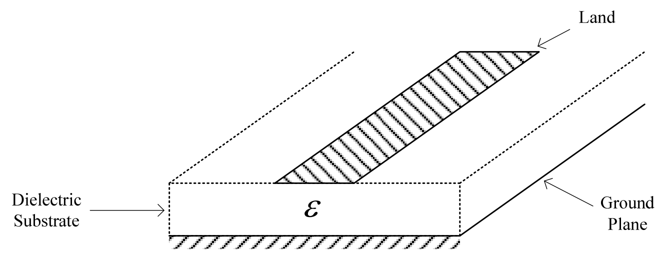
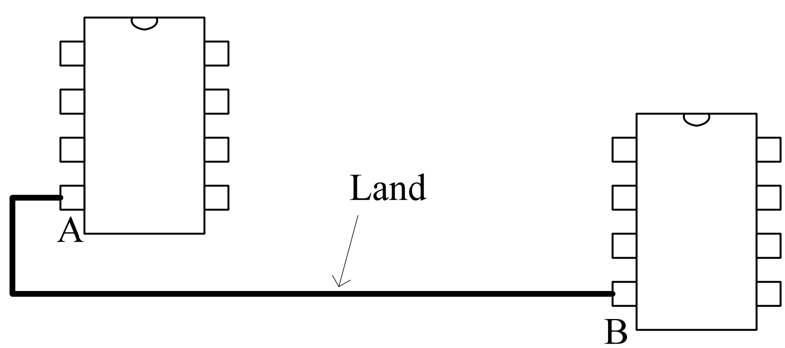
Lecture 2: Telegrapher Equations For Transmission Lines. Power Flow.

Microstrip is one method for making electrical connections in a microwave circuit. It is constructed with a ground plane on one side of a PCB and "lands" on the other:



Microstrip is an example of a transmission line, though technically it is only an approximate model for microstrip, as we will see later in this course.

Why TLs? Imagine two ICs are connected together as shown:



When the voltage at A changes state, does that new voltage appear at B instantaneously? No, of course not.

If these two points are separated by a large electrical distance, there will be a propagation delay as the change in state (electrical signal) travels to B. Not an instantaneous effect.

In microwave circuits, even distances as small as a few inches may be "far" and the propagation delay for a voltage signal to appear at another IC may be significant.

This propagation of voltage signals is modeled as a "**transmission line**" **(TL)**. We will see that voltage and current can propagate along a TL as **waves**! Fantastic.

The transmission line model can be used to solve many, many types of high frequency problems, either exactly or approximately:

- Coaxial cable.
- Two-wire.
- Microstrip, stripline, coplanar waveguide, etc.

All true TLs share one common characteristic: the $\bar{E}$ and $\bar{H}$ fields are all perpendicular to the direction of propagation, which is the long axis of the geometry. These are called TEM fields for transverse electric and magnetic fields.

An excellent example of a TL is a coaxial cable. On a TL, the voltage and current vary along the structure in time t and spatially in the z direction, as indicated in the figure below. There are no instantaneous effects.

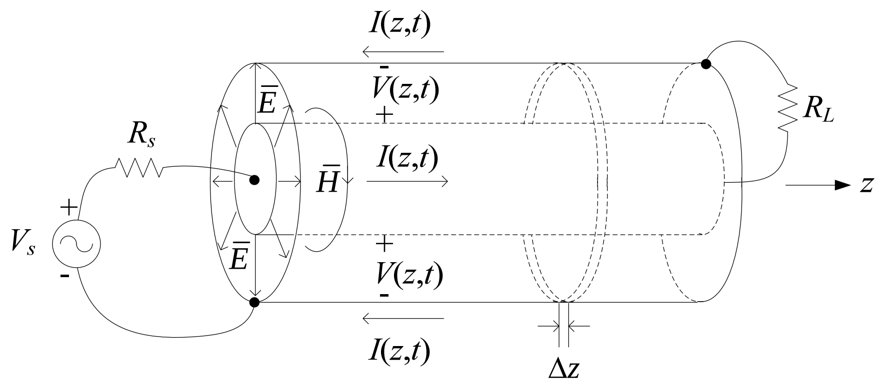
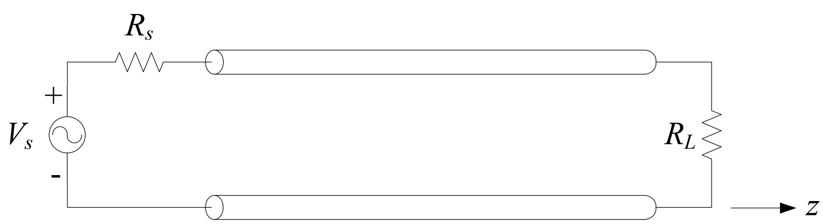


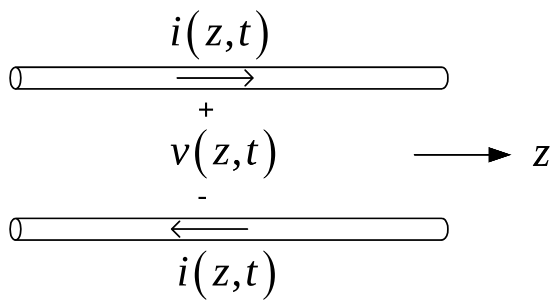
Fig. 1

A common circuit symbol for a TL is the two-wire (parallel) symbol to indicate any transmission line. For example, the equivalent circuit for the coaxial structure shown above is:



Analysis of Transmission Lines

As mentioned, on a TL the voltage and current vary along the structure in time (t) and in distance (z), as indicated in the figure above. There are no instantaneous effects.



How do we solve for $v(z,t)$ and $i(z,t)$? We first need to develop the governing equations for the voltage and current, and then solve these equations.

Notice in Fig. 1 above that there is conduction current in the center conductor and outer shield of the coaxial cable, and a displacement current between these two conductors where the electric field $\bar{E}$ is varying with time. **Each of these currents has an associated impedance**:

- Conduction current impedance effects:
  - Resistance, R, due to losses in the conductors,
  - Inductance, L, due to the current in the conductors and the magnetic flux linking the current path.
- Displacement current impedance effects:
  - Conductance, G, due to losses in the dielectric between the conductors,
  - Capacitance, C, due to the time varying electric field between the two conductors.

To develop the governing equations for $V(z,t)$ and $I(z,t)$, we will consider only a small section Δz of the TL. This Δz is so small that the electrical effects are occurring instantaneously and we can simply use circuit theory to draw the relationships between the conduction and displacement currents. This equivalent circuit is shown below:

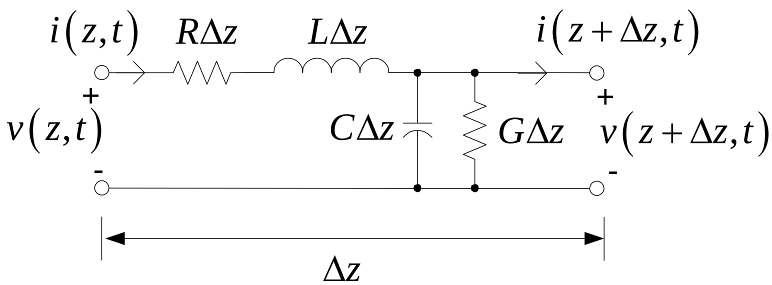
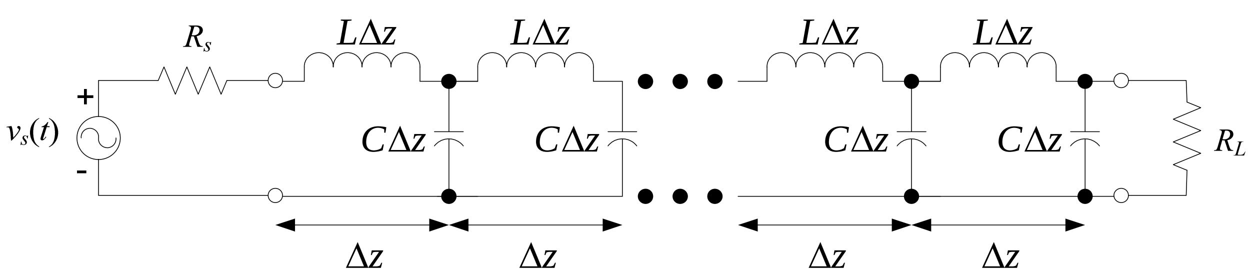


Fig. 2

The variables R, L, C, and G are distributed (or per-unit length, PUL) parameters with units of Ω/m, H/m, F/m, and S/m, respectively. We will sometimes ignore losses in this course.

A finite length of TL can be constructed by cascading many, many of these subsections along the total length of the TL. In the case of a lossless TL where $R = G = 0$ this cascade appears as:



This is a general model: it applies to any TL regardless of its cross sectional shape provided the actual electromagnetic field is TEM.

However, the PUL-parameter values change depending on the specific geometry (whether it is a microstrip, stripline, two-wire, coax, or other geometry) and the construction materials.

Transmission Line Equations

To develop the governing equation for $v(z,t)$, apply KVL in Fig. 2 above (ignoring losses)

$$v(z,t) = L\Delta z \frac{\partial i(z,t)}{\partial t} + v(z+\Delta z,t) \qquad \text{(2.1a),(1)}$$

Similarly, for the current $i(z,t)$ apply KCL at the node

$$i(z,t) = C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} + i(z+\Delta z,t) \qquad \text{(2.1b),(2)}$$

Then:

1. Divide (1) by Δz:

$$\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -L\frac{\partial i(z,t)}{\partial t} \qquad (3)$$

In the limit as $\Delta z \to 0$, the term on the LHS in (3) is the **forward difference definition** of derivative. Hence,

$$\boxed{\frac{\partial v(z,t)}{\partial z} = -L\frac{\partial i(z,t)}{\partial t}} \qquad \text{(2.2a),(4)}$$

2. Divide (2) by Δz:

$$\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z}=-C\frac{\partial v(z+\cancel{\Delta z}^{0},t)}{\partial t} \tag{5}$$

Again, in the limit as $\Delta z \to 0$ the term on the LHS is the forward difference definition of derivative. Hence,

$$\boxed{\frac{\partial i(z,t)}{\partial z}=-C\frac{\partial v(z,t)}{\partial t}} \qquad \text{(2.2b),(6)}$$

Equations (4) and (6) are a pair of coupled first order partial differential equations (PDEs) for $v(z,t)$ and $i(z,t)$. These two equations are called the telegrapher equations or the transmission line equations.

Recap: We expect that *v* and *i* are not constant along microwave circuit interconnects. Rather, (4) and (6) dictate how *v* and *i* vary along the TL at all times.

TL Wave Equations

We will now combine (4) and (6) in a special way to form two equations, each a function of *v* or *i* only.

To do this, take $\frac{\partial}{\partial z}$ of (4) and $\frac{\partial}{\partial t}$ of (6):

- $\frac{\partial}{\partial z}(4)$: $$\frac{\partial^2 v(z,t)}{\partial z^2}=-L\frac{\partial^2 i(z,t)}{\partial z \partial t} \tag{7}$$

- $\frac{\partial}{\partial t}(6)$: $$\frac{\partial^2 i(z,t)}{\partial t \partial z} = -C\frac{\partial^2 v(z,t)}{\partial t^2} \quad (8)$$

Substituting (8) into (7) gives:

$$\frac{\partial^2 v(z,t)}{\partial z^2} = LC\frac{\partial^2 v(z,t)}{\partial t^2} \quad (9)$$

Similarly,

$$\frac{\partial^2 i(z,t)}{\partial z^2} = LC\frac{\partial^2 i(z,t)}{\partial t^2} \quad (10)$$

Equations (9) and (10) are the governing equations for the *z* and *t* dependence of *v* and *i*. These are very special equations. In fact, they are wave equations for *v* and *i*!

We will define the (phase) velocity of these waveforms as

$$\boxed{v_p = \frac{1}{\sqrt{LC}} \quad \text{[m/s]}} \quad (2.16)$$

so that (9) becomes

$$\frac{\partial^2 v(z,t)}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 v(z,t)}{\partial t^2} \quad (11)$$

Voltage Wave Equation Solutions

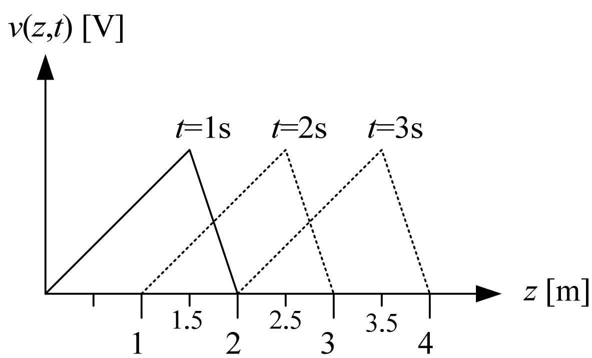
There are two general solutions to (11):

1. $v(z,t) = v_+\left(t - \frac{z}{v_p}\right)$ (12)

v_+ is any twice-differentiable function that contains t, z, and v_p in the form of the argument shown. It can be verified that (12) is a solution to (11) by substituting (12) into (11) and showing that the LHS equals the RHS.

Equation (12) represents a wave traveling in the $+z$ direction with **speed** $v_p = 1/\sqrt{LC}$ m/s.

To see this, consider the example below with v_p = 1 m/s:



At t = 1 s, focus on the peak located at z = 1.5 m. Then,

$$s_+ \equiv t - \frac{z}{v_p} = 1 - \frac{1.5}{\cancel{v_p}^{\,1}} = -0.5$$

The argument s_+ stays constant for varying t and z. Therefore, at t = 2 s, for example, then

$$s_+ = -0.5 = \cancel{t}^{\,2} - \frac{z}{\cancel{v_p}^{\,1}}$$

Therefore,

$$z = 2.5 \text{ m}$$

So the peak has now moved to position z = 2.5 m at t = 2 s.

Likewise, every point on this function moves the same distance (1 m) in this time (1 s). This is called wave motion.

The speed of this movement is

$$\frac{\Delta z}{\Delta t} = \frac{1\text{ m}}{1\text{ s}} = 1\ \frac{\text{m}}{\text{s}} = v_p$$

2. $v(z,t) = v_-\left(t + \frac{z}{v_p}\right)$ (13)

This is the second general solution to (11). This function v_- represents a wave moving in the -z direction with speed v_p.

The complete solution to the wave equation (11) is the sum of (12) and (13)

$$v(z,t) = v_+\left(t - \frac{z}{v_p}\right) + v_-\left(t + \frac{z}{v_p}\right) \tag{14}$$

v_+ and v_- can be any suitably differentiable functions, but with arguments as shown.

Current Wave Equation Solutions

A similar analysis can be performed for current waves on the TL. The governing equation for $i(z,t)$ is

$$\frac{\partial^2 i(z,t)}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 i(z,t)}{\partial t^2} \tag{15}$$

The complete general solution to this current wave equation can be determined in a manner similar to the voltage as

$$i(z,t) = \underbrace{i_+\left(t - \tfrac{z}{v_p}\right)}_{+z \text{ wave}} + \underbrace{i_-\left(t + \tfrac{z}{v_p}\right)}_{-z \text{ wave}} \tag{16}$$

Furthermore, the function i_+ can be related to the function v_+ and i_- can be related to v_-.

For example, substituting $i_+\left(t - \tfrac{z}{v_p}\right)$ and $v_+\left(t - \tfrac{z}{v_p}\right)$ into (6), differentiating then integrating gives

$$-\frac{1}{v_p} i_+ = -C v_+$$

or

$$i_+ = v_p C v_+ \tag{17}$$

But,

$$v_p C = \frac{1}{\sqrt{LC}} C = \sqrt{\frac{C}{L}}$$

We will define

$$\boxed{Z_0 \equiv \sqrt{\frac{L}{C}} \quad [\Omega]} \tag{2.13),(18}$$

as the characteristic impedance of the transmission line. (Note that in some texts, Z_0 is denoted as R_c, the characteristic resistance of the TL).

With (18), (17) can be written as

$$i_+\left(t-\tfrac{z}{v_p}\right)=\frac{v_+\left(t-\tfrac{z}{v_p}\right)}{Z_0} \tag{19}$$

Similarly, it can be shown that

$$i_-\left(t+\tfrac{z}{v_p}\right)=-\frac{v_-\left(t+\tfrac{z}{v_p}\right)}{Z_0} \tag{20}$$

The minus sign results since the current is in the -*z* direction.

Finally, substituting (19) and (20) into (16) gives

$$i(z,t)=\frac{1}{Z_0}v_+\left(t-\tfrac{z}{v_p}\right)-\frac{1}{Z_0}v_-\left(t+\tfrac{z}{v_p}\right) \tag{21}$$

This equation as well as (14)

$$v(z,t)=v_+\left(t-\tfrac{z}{v_p}\right)+v_-\left(t+\tfrac{z}{v_p}\right) \tag{22}$$

are the general wave solutions for *v* and *i* on a transmission line.

Power Flow

These voltage and current waves transport power along the TL. The power flow carried by the forward wave $p_+(z,t)$ is

$$p_+(z,t)=v_+(z,t)i_+(z,t)=\frac{v_+^2(z,t)}{Z_0} \tag{23}$$

which is positive indicating power flows in the +*z* direction. Similarly, the power flow of the reverse wave is

$$p_{-}\left(z,t\right)=v_{-}\left(z,t\right)i_{-}\left(z,t\right)=-\frac{v_{-}^{2}\left(z,t\right)}{Z_{0}} \tag{24}$$

which is negative indicating power flows in the -*z* direction.