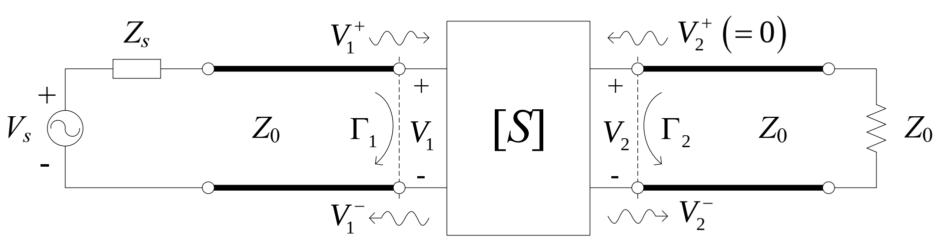
Lecture 17: *S* Parameters and Time Average Power. Generalized *S* Parameters.

There are two remaining topics concerning *S* parameters we will cover in this lecture.

The first is an important relationship between *S* parameters and relative time average power flow. The second topic is generalized scattering parameters, which are required if the port characteristic impedances are unequal.

S Parameters and Time Average Power

There is a simple and very important relationship between *S* parameters and relative time average power flow. To see this, consider a generic two-port connected to this TL circuit with a matched load:



By definition of the *S* parameters,

$$V_1^- = S_{11}V_1^+ + S_{12}V_2^+ \tag{1}$$

$$V_2^- = S_{21}V_1^+ + S_{22}V_2^+ \tag{2}$$

At port 1, the total voltage is

$$V_1 = V_1^+ + V_1^- \tag{3}$$

and the total time average power at that port is comprised of the two terms (see 2.37):

$$P_{\text{inc}} = \frac{\left|V_1^+\right|^2}{2Z_0} \quad \text{and} \quad P_{\text{ref}} = \frac{\left|V_1^-\right|^2}{2Z_0} \qquad (4),(5)$$

Further, since port 2 is matched the total voltage there is

$$V_2\Big|_{V_2^+=0} = V_2^- \tag{6}$$

Consequently, for this circuit the transmitted power is

$$P_{\text{trans}}\Big|_{V_2^+=0} = \frac{\left|V_2^-\right|^2}{2Z_0} \tag{7}$$

Using the results from (4), (5), and (7), we will consider ratios of these time average power quantities at each port and relate these ratios to the *S* parameters of the network.

- **At Port 1**. Using (4) and (5), the ratio of reflected and incident time average power is:

$$\frac{P_{\text{ref}}}{P_{\text{inc}}} = \frac{\left|V_1^-\right|^2}{\left|V_1^+\right|^2} = \left|\frac{V_1^-}{V_1^+}\right|^2 \tag{8}$$

From (1) and noticing port 2 is matched so that

$$S_{11} = \frac{V_1^-}{V_1^+}\bigg|_{V_2^+=0}$$

then in (8):

$$\left.\frac{P_{\text{ref}}}{P_{\text{inc}}}\right|_{V_2^+=0} = \left|S_{11}\right|^2 \tag{9}$$

This result teaches us that the relative reflected time average power at port 1 equals $\left|S_{11}\right|^2$ when port 2 is *matched*.

- **At Port 2**. Using (7) and (4), the ratio of transmitted and incident time average power is:

$$\frac{\left.P_{\text{trans}}\right|_{V_2^+=0}}{P_{\text{inc}}} = \frac{\left|V_2^-\right|^2}{\left|V_1^+\right|^2} \tag{10}$$

However, from (2) and with $V_2^+ = 0$, then

$$\left.\frac{P_{\text{trans}}}{P_{\text{inc}}}\right|_{V_2^+=0} = \left|S_{21}\right|^2 \tag{11}$$

This result states that the relative transmitted power to port 2 equals $\left|S_{21}\right|^2$ when port 2 is *matched*.

Equations (9) and (11) provide an extremely useful physical interpretation of the S parameters as ratios of time average power. Note that this interpretation is valid regardless of the loss (or even gain) of the network.

However, if the network is lossless we can use (9) and (11) to develop other very useful relationships. Recall that for a lossless network,

$$[S]=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

must be unitary. As a direct result of this

$$S_{11}S_{11}^* + S_{21}S_{21}^* = 1, \quad \text{or} \quad \boxed{|S_{11}|^2 + |S_{21}|^2 = 1} \tag{12a}$$

and

$$S_{12}S_{12}^* + S_{22}S_{22}^* = 1, \quad \text{or} \quad \boxed{|S_{22}|^2 + |S_{12}|^2 = 1} \tag{12b}$$

These equations are valid for all lossless two-ports.

Furthermore, in the circuit above with port 2 matched, we can additionally interpret (12a) as a conservation of power statement for the network, based on (9) and (11). If port 2 is not matched, (12a) is still valid, of course, but it is no longer a conservation of power statement for the network. **Tricky!**

Lastly, since $[S]$ is unitary, then

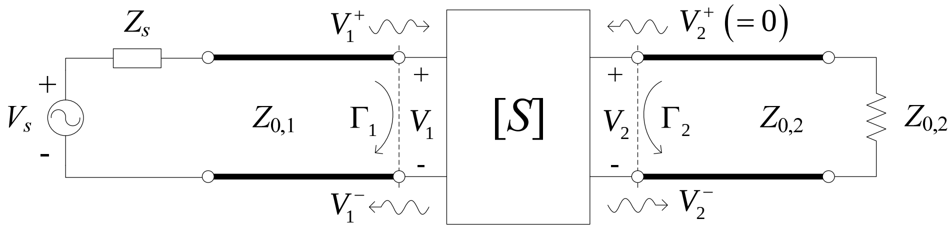
$$S_{11}S_{12}^* + S_{21}S_{22}^* = 0 \tag{13}$$

which doesn't appear to have a time average power interpretation. Can you devise a physical interpretation of (13)?

Generalized Scattering Parameters

If the characteristic impedances are different for some ports the network is connected to, it becomes necessary to redefine the scattering parameters so that $|S_{ij}|^2$ still relates to relative time average power flow.

For example, if $Z_{0,1} \neq Z_{0,2}$ in this circuit



then with port 2 matched the incident, reflected, and transmitted time average power are, respectively,

$$P_{\text{inc}} = \frac{|V_1^+|^2}{2Z_{0,1}}, \quad P_{\text{ref}} = \frac{|V_1^-|^2}{2Z_{0,1}} \qquad (14),(15)$$

and
$$P_{\text{trans}}\Big|_{V_2^+=0} = \frac{|V_2^-|^2}{2Z_{0,2}} \qquad (16)$$

Consequently,

$$\left.\frac{P_{\text{ref}}}{P_{\text{inc}}}\right|_{V_2^+=0} = \frac{|V_1^-|^2 / Z_{0,1}}{|V_1^+|^2 / Z_{0,1}} = \left|\frac{V_1^-}{V_1^+}\right|^2 = |S_{11}|^2 \qquad (17)$$

which is a familiar result, as in (9). However,

$$\left.\frac{P_{\text{trans}}}{P_{\text{inc}}}\right|_{V_2^+=0} = \frac{|V_2^-|^2 / Z_{0,2}}{|V_1^+|^2 / Z_{0,1}} \stackrel{?}{=} |S_{21}|^2 \qquad (18)$$

is not familiar, referring to (11).

To preserve the very useful interpretation of $|S_{ij}|^2$ as a relative time average power flow with matched ports, we need to redefine the S parameters when the port impedances are not equal. For example, **if** we redefine $|S_{21}|$ from (18) as

$$\left|S_{21}\right| \stackrel{?}{=} \frac{\left|V_2^-\right|}{\left|V_1^+\right|}\frac{\sqrt{Z_{0,1}}}{\sqrt{Z_{0,2}}} \tag{19}$$

then this would preserve the interpretation of time average power flow.

This redefinition leads to the so-called generalized *S* parameters. The "wave amplitude" towards port *n* is defined as

$$a_n \equiv \frac{V_n^+}{\sqrt{Z_{0,n}}} \tag{20}$$

while the "wave amplitude" away from this port is defined as

$$b_n \equiv \frac{V_n^-}{\sqrt{Z_{0,n}}} \tag{21}$$

These inward and outward going wave amplitudes are related as

$$[b] = [S]\cdot[a] \tag{22}$$

where

$$S_{ij} = \left.\frac{b_i}{a_j}\right|_{a_k=0,\forall k\neq j} \tag{23}$$

These S_{ij} are the generalized scattering parameters. They reduce to the "regular" *S* parameters when all port impedances are equal.

If we substitute (20) and (21) into (23), note that we recover (19) if $i \neq j$ and we recover (17) if $i = j$. Consequently, we can interpret the generalized scattering parameters of (22) and (23) in terms of relative reflected and transmitted time average power flows.

Lastly, at the terminal plane for port n with characteristic impedance $Z_{0,n}$, we know that the total voltage is

$$V_n = V_n^+ + V_n^- \tag{24}$$

while the current is

$$I_n = \frac{1}{Z_{0,n}}\left[V_n^+ - V_n^-\right] \tag{25}$$

Using (20) and (21) in (24) and (25), it can be shown that

$$a_n = \frac{1}{2\sqrt{Z_{0,n}}}\left(V_n + Z_{0,n} I_n\right) \tag{26}$$

$$b_n = \frac{1}{2\sqrt{Z_{0,n}}}\left(V_n - Z_{0,n} I_n\right) \tag{27}$$

We will not be using a_n, b_n, or generalized scattering parameters very much in this course. This topic is mentioned primarily to reinforce the relationship of S parameters to relative time average power and to present the "wave amplitudes" a_n and b_n, which appear widely in the literature.