

Lecture 16: Properties of S Matrices. Shifting Reference Planes.

In Lecture 14, we saw that for reciprocal networks the Z and Y matrices are:

1. Purely imaginary for lossless networks, and
2. Symmetric about the main diagonal for reciprocal networks.

In these two special instances, there are also special properties of the S matrix which we will discuss in this lecture.

Reciprocal Networks and S Matrices

In the case of **reciprocal networks**, it can be shown that

$$[S] = [S]^t \quad (4.48), (1)$$

where $[S]^t$ indicates the transpose of $[S]$. In other words, (1) is a statement that $[S]$ is symmetric about the main diagonal, which is what we also observed for the Z and Y matrices.

Lossless Networks and S Matrices

The condition for a **lossless network** is a bit more obtuse for S matrices. As derived in your text, if a network is lossless then

$$[S]^* = \{[S]^t\}^{-1} \quad (4.51),(2)$$

which, as it turns out, is a statement that $[S]$ is a **unitary matrix**.

This result can be put into a different, and possibly more useful, form by pre-multiplying (2) by $[S]^t$

$$[S]^t \cdot [S]^* = [S]^t \cdot \{[S]^t\}^{-1} = [I] \quad (3)$$

$[I]$ is the unit matrix defined as

$$[I] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Expanding (3) we obtain

$$\begin{array}{c} i \rightarrow \\ k \downarrow \end{array} \underbrace{\begin{bmatrix} S_{11} & S_{21} & \cdots & S_{N1} \\ S_{12} & S_{22} & & \vdots \\ \vdots & & \ddots & \\ S_{1N} & \cdots & & S_{NN} \end{bmatrix}}_{=[S]^t} \cdot \begin{array}{c} j \rightarrow \\ \begin{bmatrix} S_{11}^* & S_{12}^* & \cdots & S_{1N}^* \\ S_{21}^* & S_{22}^* & & \vdots \\ \vdots & & \ddots & \\ S_{N1}^* & \cdots & & S_{NN}^* \end{bmatrix} \end{array} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \quad (4)$$

Three special cases –

- Take row 1 times column 1:

$$S_{11}S_{11}^* + S_{21}S_{21}^* + \cdots + S_{N1}S_{N1}^* = 1 \quad (5)$$

Generalizing this result gives

$$\sum_{k=1}^N S_{ki} S_{ki}^* = 1 \quad i = 1, \dots, N \quad (4.53a), (6)$$

In words, this result states that the dot product of any column of $[S]$ with the conjugate of that same column **equals 1** (for a lossless network).

- Take row 1 times column 2:

$$S_{11} S_{12}^* + S_{21} S_{22}^* + \dots + S_{N1} S_{N2}^* = 0$$

Generalizing this result gives

$$\sum_{k=1}^N S_{ki} S_{kj}^* = 0 \quad \forall (i, j), i \neq j \quad (4.53b), (7)$$

In words, this result states that the dot product of any column of $[S]$ with the conjugate of another column **equals 0** (for a lossless network).

- Applying (1) to (7): If the network is also reciprocal, then $[S]$ is symmetric and we can make a similar statement concerning the rows of $[S]$.

That is, the dot product of any row of $[S]$ with the conjugate of another row **equals 0** (for a lossless, reciprocal network).

Example N16.1 In a homework assignment, the S matrix of a two port network was given to be

$$[S] = \begin{bmatrix} 0.2 + j0.4 & 0.8 - j0.4 \\ 0.8 - j0.4 & 0.2 + j0.4 \end{bmatrix}$$

Is the network reciprocal? **Yes**, because $[S]^t = [S]$.

Is the network lossless? This question often cannot be answered simply by quick inspection of the S matrix.

Rather, we will systematically apply the conditions stated above to the columns of the S matrix:

- $C1 \cdot C1^*$: $(0.2 + j0.4)(0.2 - j0.4) + (0.8 - j0.4)(0.8 + j0.4) = 1$
- $C2 \cdot C2^*$: Same = 1
- $C1 \cdot C2^*$: $(0.2 + j0.4)(0.8 + j0.4) + (0.8 - j0.4)(0.2 - j0.4) = 0$
- $C2 \cdot C1^*$: Same = 0

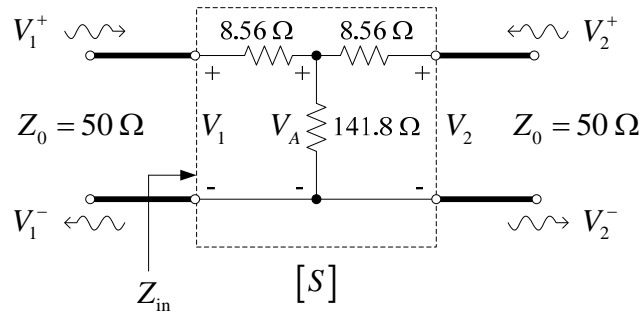
Therefore, the network is **lossless**.

As an aside, in Example N15.1 of the text, which we saw in the last lecture,

$$[S] = \begin{bmatrix} 0.1 & j0.8 \\ j0.8 & 0.2 \end{bmatrix}$$

This network is obviously reciprocal, and it can be shown that it's also lossy. (Go ahead, give it a try.)

Example N16.2 (Text Example 4.4). Determine the S parameters for this T -network assuming a $50\text{-}\Omega$ system impedance, as shown.



First, take a *general* look at the circuit:

- It's linear, so it must also be **reciprocal**. Consequently, $[S]$ must be symmetrical (about the main diagonal).
- The circuit appears **unchanged** when “viewed” from either port 1 or port 2. Consequently, $S_{11} = S_{22}$.

Based on these observations, we only need to determine S_{11} and S_{21} since $S_{22} = S_{11}$ and $S_{12} = S_{21}$.

Proceeding, recall that S_{11} is the reflection coefficient at port 1 with port 2 matched:

$$S_{11} \equiv \left. \frac{V_1^-}{V_1^+} \right|_{V_2^+ = 0} = \Gamma_{11} \Big|_{V_2^+ = 0}$$

The input impedance with port 2 matched is

$$Z_{\text{in}} = 8.56 + 141.8 \parallel (8.56 + 50) \Omega = 50.00 \Omega$$

which (it will turn out not coincidentally) equals Z_0 ! With this Z_{in} :

$$S_{11} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = 0$$

which also implies $S_{22} = 0$.

Next, for S_{21} we apply a V_1^+ with port 2 matched and measure V_2^- :

$$S_{21} = \left. \frac{V_2^-}{V_1^+} \right|_{V_2^+ = 0}$$

At port 1, which we will also assume is the terminal plane, $V_1 = V_1^+ + V_1^-$. However, with $50\text{-}\Omega$ termination at port 2, $V_1^- = 0$ (since $\Gamma_{11} = 0$). Therefore, $V_1 = V_1^+$. Similarly, $V_2 = V_2^-$.

These last findings imply we can simply use **voltage division** to determine V_2^- (from V_2):

$$V_A = \frac{141.8 \parallel (50 + 8.56)}{141.8 \parallel (50 + 8.56) + 8.56} \cdot V_1 = 0.8288V_1$$

and
$$V_2 = \frac{50}{50 + 8.56} \cdot V_A = 0.8538 \cdot 0.8288V_1 = 0.7077V_1$$

Therefore,
$$V_2^- = 0.7077V_1^+ \Rightarrow S_{21} = \frac{1}{\sqrt{2}} = S_{12}$$

The complete S matrix for the given T -network referenced to $50\text{-}\Omega$ system impedance is therefore

$$[S] = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Lastly, notice that when port 2 is matched

$$S_{21} = \frac{1}{\sqrt{2}} = T_{21}|_{V_2^+=0}$$

so that

$$\left| T_{21}|_{V_2^+=0} \right|^2 = \frac{1}{2}$$

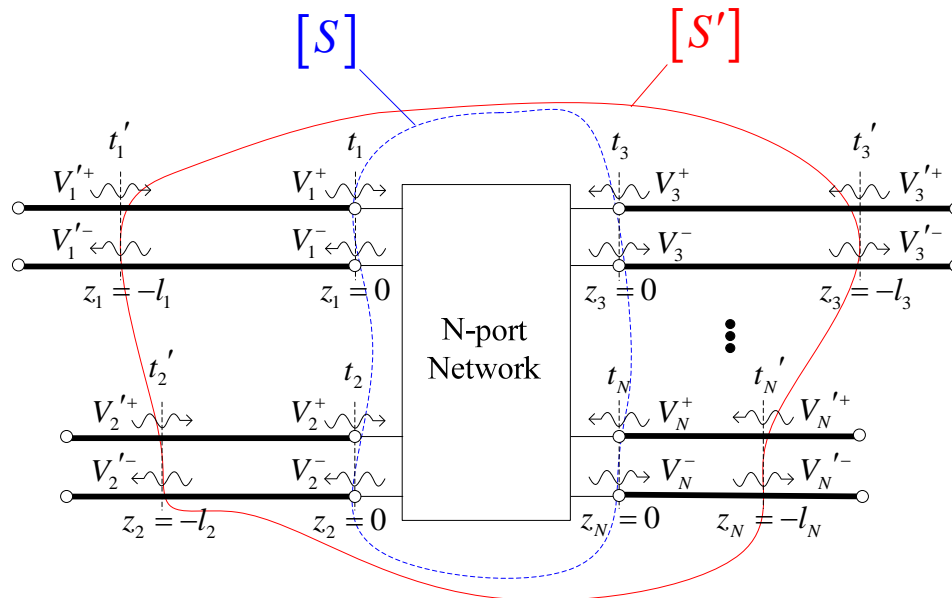
which says that half of the power incident from port 1 is transmitted to port 2 when port 2 is matched. We can now see why this T -network is called a **3-dB attenuator**.

Shifting Reference Planes

Recall that when we defined S parameters for a network, terminal planes were defined for all ports. These are arbitrarily chosen phase = 0° locations on TLs connected to the ports.

It turns out that S parameters change very simply and predictably as the reference planes are varied along lossless TLs. This fact can prove handy, especially in the lab.

Referring to Fig 4.9:



To be specific, let $[S]$ be the scattering matrix of a network with reference planes (i.e., ports) at t_n , while $[S']$ is the scattering matrix of the network with the reference planes shifted to t_n' .

Applying the definition of the scattering matrix in these two situations yields

$$[V^-] = [S] \cdot [V^+] \quad (4.54a), (8)$$

and

$$[V'^-] = [S'] \cdot [V'^+] \quad (4.54b), (9)$$

We've shifted the reference planes along lossless TLs. Hence, these voltage amplitudes only change phase as

$$V_n'^+ = V_n^+ e^{+j\theta_n} \quad (4.55a), (10)$$

and

$$V_n'^- = V_n^- e^{-j\theta_n} \quad (4.55b), (11)$$

where $\theta_n = \beta_n l_n$. Remember, these are the phase shifts when the phase planes are all moved **away** from the ports.

It is easy to prove these phase shift relationships in (10) and (11). First, we know that $V_n^+(z_n) = V_n^+ e^{-j\beta_n z_n}$. Hence, $V_n^+(-l_n) = V_n^+ e^{+j\beta_n l_n}$. Therefore, $V_n'^+ \equiv V_n^+(-l_n) = V_n^+ e^{+j\theta_n}$, which is (10).

Likewise, $V_n^-(z_n) = V_n^- e^{j\beta_n z_n}$ so that $V_n^-(-l_n) = V_n^- e^{-j\beta_n l_n}$. Therefore, $V_n'^- \equiv V_n^-(-l_n) = V_n^- e^{-j\theta_n}$, which is (11).

Now, armed only with this information in (10) and (11), we can **express $[S']$ in terms of $[S]$** . Writing (10) and (11) in matrix form and substituting these into (8)

$$[V^-] = [S] \cdot [V^+] \quad (8)$$

gives:

$$\begin{bmatrix} e^{j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{j\theta_N} \end{bmatrix} \cdot [V'^-] = [S] \cdot \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \cdot [V'^+] \quad (12)$$

The inverse of a diagonal matrix is simply a diagonal matrix with inverted diagonal elements. So, we can pre-multiply (12) by the inverse of the first matrix (which is known, and is also not singular) giving:

$$[V'^-] = \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \cdot [S] \cdot \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \cdot [V'^+] \quad (13)$$

Comparing (13) with (9) we can immediately deduce that:

$$[S'] = \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \cdot [S] \cdot \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \quad (4.56),(14)$$

Multiplying out this matrix equation gives:

$$S'_{mn} = S_{mn} e^{-j(\theta_m + \theta_n)} \quad (15)$$

and when $m = n$,

$$S'_{nn} = S_{nn} e^{-j2\theta_n} \quad (16)$$

The factor of two in this last exponent arises since the wave travels twice the electrical distance θ_n : once towards the port and once back to the new terminal plane t'_n .

Equations (15) and (16) provide the **simple transformations** for S parameters when the phase planes are shifted away from the ports.

Many times you'll find that your measured S parameters differ from simulation by a phase angle, even though the magnitude is in good agreement. This likely occurred because your **terminal planes were defined differently** in your simulations than was set during measurement.