Lecture 16: Properties of *S* Matrices. Shifting Reference Planes.

In Lecture 14, we saw that for reciprocal networks the *Z* and *Y* matrices are:

1. Purely imaginary for lossless networks, and
2. Symmetric about the main diagonal for reciprocal networks.

In these two special instances, there are also special properties of the *S* matrix which we will discuss in this lecture.

Reciprocal Networks and *S* Matrices

In the case of reciprocal networks, it can be shown that

$$\boxed{[S]=[S]^t} \qquad (4.48),(1)$$

where $[S]^t$ indicates the transpose of $[S]$. In other words, (1) is a statement that $[S]$ is symmetric about the main diagonal, which is what we also observed for the *Z* and *Y* matrices.

Lossless Networks and *S* Matrices

The condition for a lossless network is a bit more obtuse for *S* matrices. As derived in your text, if a network is lossless then

$$[S]^* = \left\{[S]^t\right\}^{-1} \qquad (4.51),(2)$$

which, as it turns out, is a statement that $[S]$ is a unitary matrix.

This result can be put into a different, and possibly more useful, form by pre-multiplying (2) by $[S]^t$

$$[S]^t \cdot [S]^* = [S]^t \cdot \left\{[S]^t\right\}^{-1} = [I] \qquad (3)$$

$[I]$ is the unit matrix defined as

$$[I] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Expanding (3) we obtain

$$\begin{array}{cc} & i \rightarrow \\ k \downarrow & \underbrace{\begin{bmatrix} S_{11} & S_{21} & \cdots & S_{N1} \\ S_{12} & S_{22} & & \vdots \\ \vdots & & \ddots & \\ S_{1N} & \cdots & & S_{NN} \end{bmatrix}}_{=[S]^t} \end{array} \cdot \begin{array}{c} j \rightarrow \\ \begin{bmatrix} S_{11}^* & S_{12}^* & \cdots & S_{1N}^* \\ S_{21}^* & S_{22}^* & & \vdots \\ \vdots & & \ddots & \\ S_{N1}^* & \cdots & & S_{NN}^* \end{bmatrix} \end{array} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \qquad (4)$$

Three special cases –

- Take row 1 times column 1:

$$S_{11}S_{11}^* + S_{21}S_{21}^* + \cdots + S_{N1}S_{N1}^* = 1 \qquad (5)$$

Generalizing this result gives

$$\boxed{\sum_{k=1}^{N} S_{ki} S_{ki}^{*} = 1 \quad i = 1, \ldots, N} \tag{4.53a),(6}$$

In words, this result states that the dot product of any column of $[S]$ with the conjugate of that same column equals 1 (for a lossless network).

- Take row 1 times column 2:

$$S_{11}S_{12}^{*} + S_{21}S_{22}^{*} + \cdots + S_{N1}S_{N2}^{*} = 0$$

Generalizing this result gives

$$\boxed{\sum_{k=1}^{N} S_{ki} S_{kj}^{*} = 0 \quad \forall (i, j), i \neq j} \tag{4.53b),(7}$$

In words, this result states that the dot product of any column of $[S]$ with the conjugate of another column equals 0 (for a lossless network).

- Applying (1) to (7): If the network is also reciprocal, then $[S]$ is symmetric and we can make a similar statement concerning the rows of $[S]$.

  That is, the dot product of any row of $[S]$ with the conjugate of another row equals 0 (for a lossless, reciprocal network).

Example N16.1 In a homework assignment, the S matrix of a two port network was given to be

$$[S]=\begin{bmatrix} 0.2+j0.4 & 0.8-j0.4 \\ 0.8-j0.4 & 0.2+j0.4 \end{bmatrix}$$

Is the network reciprocal? Yes, because $[S]^t=[S]$.

Is the network lossless? This question often cannot be answered simply by quick inspection of the *S* matrix.

Rather, we will systematically apply the conditions stated above to the columns of the *S* matrix:

- $\mathrm{C1}\cdot\mathrm{C1}^*$: $(0.2+j0.4)(0.2-j0.4)+(0.8-j0.4)(0.8+j0.4)=1$
- $\mathrm{C2}\cdot\mathrm{C2}^*$: Same = 1
- $\mathrm{C1}\cdot\mathrm{C2}^*$: $(0.2+j0.4)(0.8+j0.4)+(0.8-j0.4)(0.2-j0.4)=0$
- $\mathrm{C2}\cdot\mathrm{C1}^*$: Same = 0

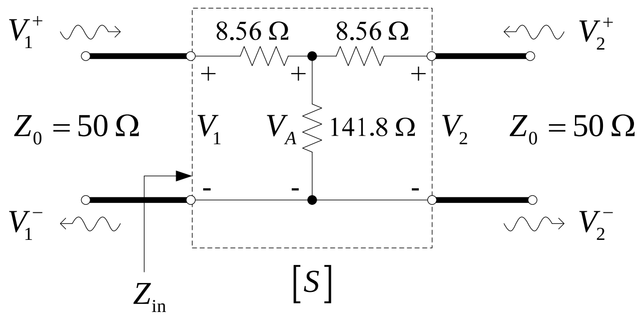
Therefore, the network is lossless.

As an aside, in Example N15.1 of the text, which we saw in the last lecture,

$$[S]=\begin{bmatrix} 0.1 & j0.8 \\ j0.8 & 0.2 \end{bmatrix}$$

This network is obviously reciprocal, and it can be shown that it's also lossy. (Go ahead, give it a try.)

Example N16.2 (Text Example 4.4). Determine the *S* parameters for this *T*-network assuming a 50-Ω system impedance, as shown.



First, take a *general* look at the circuit:

- It's linear, so it must also be reciprocal. Consequently, $[S]$ must be symmetrical (about the main diagonal).
- The circuit appears unchanged when "viewed" from either port 1 or port 2. Consequently, $S_{11} = S_{22}$.

Based on these observations, we only need to determine S_{11} and S_{21} since $S_{22} = S_{11}$ and $S_{12} = S_{21}$.

Proceeding, recall that S_{11} is the reflection coefficient at port 1 with port 2 matched:

$$S_{11} \equiv \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0} = \left.\Gamma_{11}\right|_{V_2^+=0}$$

The input impedance with port 2 matched is

$$Z_{\text{in}} = 8.56 + 141.8 \| (8.56 + 50)\ \Omega = 50.00\ \Omega$$

which (it will turn out not coincidentally) equals Z_0! With this Z_{in}:

$$S_{11} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = 0$$

which also implies $S_{22} = 0$.

Next, for S_{21} we apply a V_1^+ with port 2 matched and measure V_2^-:

$$S_{21} = \left. \frac{V_2^-}{V_1^+} \right|_{V_2^+=0}$$

At port 1, which we will also assume is the terminal plane, $V_1 = V_1^+ + V_1^-$. However, with 50-Ω termination at port 2, $V_1^- = 0$ (since $\Gamma_{11} = 0$). Therefore, $V_1 = V_1^+$. Similarly, $V_2 = V_2^-$.

These last findings imply we can simply use voltage division to determine V_2^- (from V_2):

$$V_A = \frac{141.8 \| (50 + 8.56)}{141.8 \| (50 + 8.56) + 8.56} \cdot V_1 = 0.8288 V_1$$

and $$V_2 = \frac{50}{50 + 8.56} \cdot V_A = 0.8538 \cdot 0.8288 V_1 = 0.7077 V_1$$

Therefore, $$V_2^- = 0.7077 V_1^+ \quad \Rightarrow \quad S_{21} = \frac{1}{\sqrt{2}} = S_{12}$$

The complete S matrix for the given T-network referenced to 50-Ω system impedance is therefore

$$\boxed{[S] = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}}$$

Lastly, notice that when port 2 is matched

$$S_{21} = \frac{1}{\sqrt{2}} = T_{21}\Big|_{V_2^+=0}$$

so that
$$\left| T_{21}\Big|_{V_2^+=0} \right|^2 = \frac{1}{2}$$

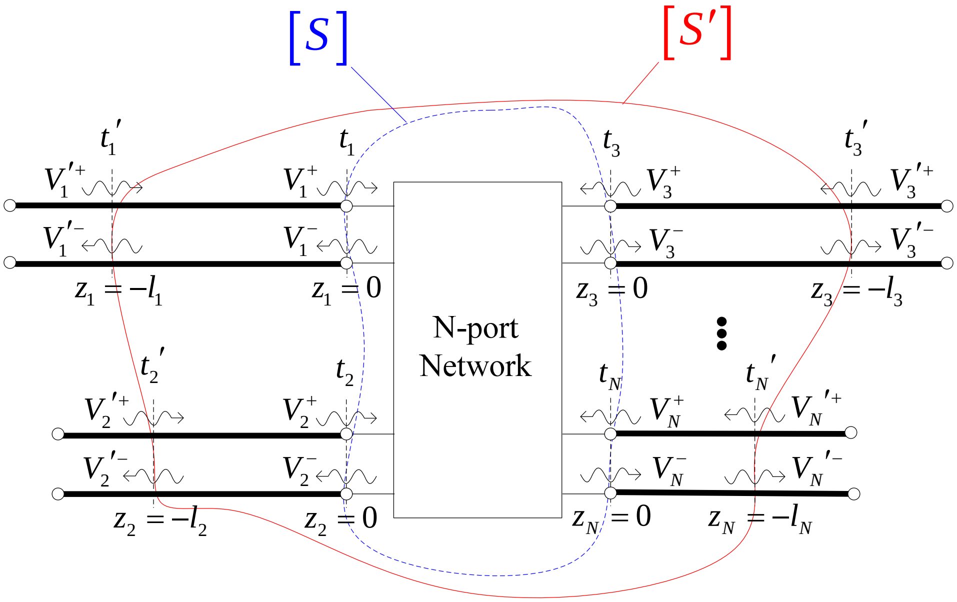
which says that half of the power incident from port 1 is transmitted to port 2 when port 2 is matched. We can now see why this *T*-network is called a 3-dB attenuator.

Shifting Reference Planes

Recall that when we defined *S* parameters for a network, terminal planes were defined for all ports. These are arbitrarily chosen phase $= 0^\circ$ locations on TLs connected to the ports.

It turns out that *S* parameters change very simply and predictably as the reference planes are varied along lossless TLs. This fact can prove handy, especially in the lab.

Referring to Fig 4.9:



To be specific, let $[S]$ be the scattering matrix of a network with reference planes (i.e., ports) at t_n, while $[S']$ is the scattering matrix of the network with the reference planes shifted to t_n'.

Applying the definition of the scattering matrix in these two situations yields

$$[V^-] = [S] \cdot [V^+] \tag{(4.54a),(8)}$$

and

$$[V'^-] = [S'] \cdot [V'^+] \tag{(4.54b),(9)}$$

We've shifted the reference planes along lossless TLs. Hence, these voltage amplitudes only change phase as

$$\boxed{V_n'^+ = V_n^+ e^{+j\theta_n}} \tag{(4.55a),(10)}$$

and

$$\boxed{V_n'^- = V_n^- e^{-j\theta_n}} \tag{(4.55b),(11)}$$

where $\theta_n = \beta_n l_n$. Remember, these are the phase shifts when the phase planes are all moved **away** from the ports.

It is easy to prove these phase shift relationships in (10) and (11). First, we know that $V_n^+(z_n) = V_n^+ e^{-j\beta_n z_n}$. Hence, $V_n^+(-l_n) = V_n^+ e^{+j\beta_n l_n}$. Therefore, $V_n'^+ \equiv V_n^+(-l_n) = V_n^+ e^{+j\theta_n}$, which is (10).

Likewise, $V_n^-(z_n) = V_n^- e^{j\beta_n z_n}$ so that $V_n^-(-l_n) = V_n^- e^{-j\beta_n l_n}$. Therefore, $V_n'^- \equiv V_n^-(-l_n) = V_n^- e^{-j\theta_n}$, which is (11).

Now, armed only with this information in (10) and (11), we can express $[S']$ in terms of $[S]$. Writing (10) and (11) in matrix form and substituting these into (8)

$$[V^-] = [S] \cdot [V^+] \tag{8}$$

gives:

$$\begin{bmatrix} e^{j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{j\theta_N} \end{bmatrix} \cdot [V'^-] = [S] \cdot \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \cdot [V'^+] \tag{12}$$

The inverse of a diagonal matrix is simply a diagonal matrix with inverted diagonal elements. So, we can pre-multiply (12) by the inverse of the first matrix (which is known, and is also not singular) giving:

$$[V'^-] = \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \cdot [S] \cdot \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \cdot [V'^+] \tag{13}$$

Comparing (13) with (9) we can immediately deduce that:

$$[S'] = \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \cdot [S] \cdot \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} \qquad (4.56),(14)$$

Multiplying out this matrix equation gives:

$$\boxed{S_{mn}' = S_{mn} e^{-j(\theta_m + \theta_n)}} \qquad (15)$$

and when $m = n$,

$$\boxed{S_{nn}' = S_{nn} e^{-j2\theta_n}} \qquad (16)$$

The factor of two in this last exponent arises since the wave travels twice the electrical distance θ_n: once towards the port and once back to the new terminal plane t_n'.

Equations (15) and (16) provide the simple transformations for *S* parameters when the phase planes are shifted away from the ports.

Many times you'll find that your measured *S* parameters differ from simulation by a phase angle, even though the magnitude is in good agreement. This likely occurred because your terminal planes were defined differently in your simulations than was set during measurement.