
In Lecture 14, we saw that for reciprocal networks the $Z$ and $Y$ matrices are:

1. Purely imaginary for lossless networks, and
2. Symmetric about the main diagonal for reciprocal networks.

In these two special instances, there are also special properties of the $S$ matrix which we will discuss in this lecture.

Reciprocal Networks and $S$ Matrices

In the case of reciprocal networks, it can be shown that

$$[S] = [S]^T \ (4.48),(1)$$

where $[S]^T$ indicates the transpose of $[S]$. In other words, (1) is a statement that $[S]$ is symmetric about the main diagonal, which is what we also observed for the $Z$ and $Y$ matrices.

Lossless Networks and $S$ Matrices

The condition for a lossless network is a bit more obtuse for $S$ matrices. As derived in your text, if a network is lossless then
\[ [S]^* = \left\{ [S]^t \right\}^{-1} \]  
(4.51), (2)

which, as it turns out, is a statement that \([S]\) is a unitary matrix.

This result can be put into a different, and possibly more useful, form by pre-multiplying (2) by \([S]^t\)

\[ [S]^t \cdot [S]^* = [S]^t \cdot \left\{ [S]^t \right\}^{-1} = [I] \]  
(3)

\([I]\) is the unit matrix defined as

\[ [I] = \begin{bmatrix} 1 & 0 \\ \vdots & \ddots \\ 0 & 1 \end{bmatrix} \]

Expanding (3) we obtain

\[
\begin{bmatrix}
S_{11} & S_{21} & \cdots & S_{N1} \\
S_{12} & S_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
S_{1N} & \cdots & S_{NN}
\end{bmatrix}
\begin{bmatrix}
S_{11}^* & S_{12}^* & \cdots & S_{1N}^* \\
S_{21}^* & S_{22}^* & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
S_{N1}^* & \cdots & S_{NN}^*
\end{bmatrix}
= \begin{bmatrix} 1 & 0 \\ \vdots & \ddots \\ 0 & 1 \end{bmatrix}
\]

(4)

Three special cases –

• Take row 1 times column 1:

\[
S_{11}S_{11}^* + S_{21}S_{21}^* + \cdots + S_{N1}S_{N1}^* = 1
\]

(5)

Generalizing this result gives
\[ \sum_{k=1}^{N} S_{ki}S_{ki}^* = 1 \quad i = 1, \ldots, N \quad (4.53a),(6) \]

In words, this result states that the dot product of any column of \([S]\) with the conjugate of that same column equals 1 (for a lossless network).

- Take row 1 times column 2:
  \[ S_{11}S_{12}^* + S_{21}S_{22}^* + \cdots + S_{N1}S_{N2}^* = 0 \]

Generalizing this result gives
\[ \sum_{k=1}^{N} S_{ki}S_{kj}^* = 0 \quad \forall (i, j), i \neq j \quad (4.53b),(7) \]

In words, this result states that the dot product of any column of \([S]\) with the conjugate of another column equals 0 (for a lossless network).

- Applying (1) to (7): If the network is also reciprocal, then \([S]\) is symmetric and we can make a similar statement concerning the rows of \([S]\).

That is, the dot product of any row of \([S]\) with the conjugate of another row equals 0 (for a lossless, reciprocal network).

---

**Example N16.1** In a homework assignment, the \(S\) matrix of a two port network was given to be
\[
[S] = \begin{bmatrix}
0.2 + j0.4 & 0.8 - j0.4 \\
0.8 - j0.4 & 0.2 + j0.4 \\
\end{bmatrix}
\]

Is the network reciprocal? Yes, because \([S]^T = [S]\).

Is the network lossless? This question often cannot be answered simply by quick inspection of the \(S\) matrix.

Rather, we will systematically apply the conditions stated above to the columns of the \(S\) matrix:

- \(C_1 \cdot C_1^*\): \((0.2 + j0.4)(0.2 - j0.4) + (0.8 - j0.4)(0.8 + j0.4) = 1\)
- \(C_2 \cdot C_2^*\): Same = 1
- \(C_1 \cdot C_2^\ast\): \((0.2 + j0.4)(0.8 + j0.4) + (0.8 - j0.4)(0.2 - j0.4) = 0\)
- \(C_2 \cdot C_1^\ast\): Same = 0

Therefore, the network is \textbf{lossless}.

As an aside, in Example N15.1 of the text, which we saw in the last lecture,

\[
[S] = \begin{bmatrix}
0.1 & j0.8 \\
j0.8 & 0.2 \\
\end{bmatrix}
\]

This network is obviously reciprocal, and it can be shown that it’s also lossy. (Go ahead, give it a try.)

\textbf{Example N16.2 (Text Example 4.4).} Determine the \(S\) parameters for this \(T\)-network assuming a 50-\(\Omega\) system impedance, as shown.
First, take a *general* look at the circuit:

- It’s linear, so it must also be *reciprocal*. Consequently, $[S]$ must be symmetrical (about the main diagonal).
- The circuit appears *unchanged* when “viewed” from either port 1 or port 2. Consequently, $S_{11} = S_{22}$.

Based on these observations, we only need to determine $S_{11}$ and $S_{21}$ since $S_{22} = S_{11}$ and $S_{12} = S_{21}$.

Proceeding, recall that $S_{11}$ is the reflection coefficient at port 1 with port 2 matched:

$$S_{11} = \left. \frac{V^-}{V^+} \right|_{V_2^+ = 0} = \Gamma_{11}$$

The input impedance with port 2 matched is

$$Z_{\text{in}} = 8.56 + 141.8 \| (8.56 + 50) \Omega = 50.00 \Omega$$

which (it will turn out not coincidentally) equals $Z_0$! With this $Z_{\text{in}}$:

$$S_{11} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = 0$$

which also implies $S_{22} = 0$. 
Next, for $S_{21}$ we apply a $V_1^+$ with port 2 matched and measure $V_2^-$:

$$S_{21} = \frac{V_2^-}{V_1^+} \bigg|_{V_2^+ = 0}$$

At port 1, which we will also assume is the terminal plane, $V_1 = V_1^+ + V_1^-$. However, with 50-Ω termination at port 2, $V_1^- = 0$ (since $\Gamma_{11} = 0$). Therefore, $V_1 = V_1^+$. Similarly, $V_2 = V_2^-$. These last findings imply we can simply use voltage division to determine $V_2^-$ (from $V_2$):

$$V_A = \frac{141.8 \cdot (50 + 8.56)}{141.8 \cdot (50 + 8.56) + 8.56} \cdot V_1 = 0.8288 V_1$$

and

$$V_2 = \frac{50}{50 + 8.56} \cdot V_A = 0.8538 \cdot 0.8288 V_1 = 0.7077 V_1$$

Therefore, $V_2^- = 0.7077 V_1^+ \implies S_{21} = \frac{1}{\sqrt{2}} = S_{12}$

The complete $S$ matrix for the given $T$-network referenced to 50-Ω system impedance is therefore

$$[S] = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Lastly, notice that when port 2 is matched
\[ S_{21} = \frac{1}{\sqrt{2}} = T_{21}\big|_{V_2^+ = 0} \]

so that

\[ \left| T_{21}\big|_{V_2^+ = 0} \right|^2 = \frac{1}{2} \]

which says that half of the power incident from port 1 is transmitted to port 2 when port 2 is matched. We can now see why this T-network is called a 3-dB attenuator.

---

**Shifting Reference Planes**

Recall that when we defined \( S \) parameters for a network, terminal planes were defined for all ports. These are arbitrarily chosen phase = 0° locations on TLs connected to the ports.

It turns out that \( S \) parameters change very simply and predictably as the reference planes are varied along lossless TLs. This fact can prove handy, especially in the lab.

Referring to Fig 4.9:
To be specific, let $[S]$ be the scattering matrix of a network with reference planes (i.e., ports) at $t_n$, while $[S']$ is the scattering matrix of the network with the reference planes shifted to $t'_n$.

Applying the definition of the scattering matrix in these two situations yields

$$[V^-] = [S] \cdot [V^+] \tag{4.54a),(8}$$

and

$$[V'^{-}] = [S'] \cdot [V'^+] \tag{4.54b),(9}$$

We’ve shifted the reference planes along lossless TLs. Hence, these voltage amplitudes only change phase as

$$V'^+ = V_n^+ e^{j \theta_n} \tag{4.55a),(10}$$

and

$$V'^- = V_n^- e^{-j \theta_n} \tag{4.55b),(11}$$

where $\theta_n = \beta_n l_n$. Remember, these are the phase shifts when the phase planes are all moved away from the ports.
It is easy to prove these phase shift relationships in (10) and (11). First, we know that \( V_n^+(z_n) = V_n^+ e^{-j\beta_n z_n} \). Hence, \( V_n^+(-l_n) = V_n^+ e^{+j\beta_n l_n} \). Therefore, \( V_n^{'+} \equiv V_n^+(-l_n) = V_n^+ e^{+j\theta_n} \), which is (10).

Likewise, \( V_n^-(z_n) = V_n^- e^{j\beta_n z_n} \) so that \( V_n^-(l_n) = V_n^- e^{+j\beta_n l_n} \). Therefore, \( V_n^{'-' \equiv} V_n^-(-l_n) = V_n^- e^{-j\theta_n} \), which is (11).

Now, armed only with this information in (10) and (11), we can express \([S']\) in terms of \([S]\). Writing (10) and (11) in matrix form and substituting these into (8)

\[
[V^-] = [S] \cdot [V^+]
\]

(8)

gives:

\[
\begin{bmatrix}
e^{j\theta_1} & 0 \\
. & . \\
0 & e^{j\theta_N}
\end{bmatrix} \cdot [V^{'\cdot}] = [S] \cdot \begin{bmatrix}
e^{-j\theta_1} & 0 \\
. & . \\
0 & e^{-j\theta_N}
\end{bmatrix} \cdot [V^{'+}] \\
(12)
\]

The inverse of a diagonal matrix is simply a diagonal matrix with inverted diagonal elements. So, we can pre-multiply (12) by the inverse of the first matrix (which is known, and is also not singular) giving:

\[
[V^{'\cdot}] = \begin{bmatrix}
e^{-j\theta_1} & 0 \\
. & . \\
0 & e^{-j\theta_N}
\end{bmatrix} \cdot [S] \cdot \begin{bmatrix}
e^{-j\theta_1} & 0 \\
. & . \\
0 & e^{-j\theta_N}
\end{bmatrix} \cdot [V^{'+}] \\
(13)
\]

Comparing (13) with (9) we can immediately deduce that:
\[
[S'] = \begin{bmatrix}
e^{-j\theta_1} & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & e^{-j\theta_N}
\end{bmatrix} \cdot [S] \cdot \begin{bmatrix}
e^{-j\theta_1} & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & e^{-j\theta_N}
\end{bmatrix}
\] (4.56), (14)

Multiplying out this matrix equation gives:
\[
S'_{mn} = S_{mn}e^{-j(\theta_m + \theta_n)}
\] (15)

and when \( m = n \),
\[
S'_{nn} = S_{nn}e^{-j2\theta_n}
\] (16)

The factor of two in this last exponent arises since the wave travels twice the electrical distance \( \theta_n \): once towards the port and once back to the new terminal plane \( t_n' \).

Equations (15) and (16) provide the simple transformations for \( S \) parameters when the phase planes are shifted away from the ports.

Many times you’ll find that your measured \( S \) parameters differ from simulation by a phase angle, even though the magnitude is in good agreement. This likely occurred because your terminal planes were defined differently in your simulations than was set during measurement.